Branching random walks with selection

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Marches aléatoires avec branchement et sélection

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Marches aléatoires avec branchement et sélection

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Résumé

Nous considérons le mouvement brownien branchant qui est un objet mathématique modélisant l'évolution d'une population. Dans ce système, les individus se déplacent indépendamment selon des mouvement browniens et se divisent indépendamment à taux 1 en deux individus. Nous nous intéressons à la position la plus à droite (resp. à gauche) au temps s, qui est définie comme le maximum (resp. le minimum) des positions des individus vivants à ce temps-là. D'après Lalley et Sellke [101], chaque individu apparu dans ce système aura un descendant atteignant la position la plus à droite. Nous étudions ce phénomène quantitativement, en estimant le premier instant où chaque individu vivant à l'instant s a eu un tel descendant.

Nous étudions ensuite la marche aléatoire branchante en temps discret qui est un système analogue dans lequel les marches aléatoires sont indexées par un arbre de Galton-Watson. On définit de la même façon la position la plus à droite et celle la plus à gauche à la génération n. Nous considérons le chemin reliant la racine à la position la plus à gauche. Nous montrons que cette marche, convenablement renormalisée, converge en loi vers une excursion brownienne normalisée.

Dans la dernière partie de cette thèse, nous nous plaçons "dans un cadre avec un critère de sélection". Étant donné un arbre régulier dont chaque individu a N enfants, nous attachons à chaque individu une variable aléatoire. Toutes les variables attachées sont i.i.d., de loi uniforme sur $[0,1]$. La sélection intervient de la façon suivante : un individu est conservé si le long du chemin le plus court le reliant à la racine, les variables aléatoires attachées sont croissantes ; les autres individus sont éliminés du système. Nous étudions le comportement asymptotique de la population dans le processus lorsque N tend vers l'infini.

Mots-clefs : Marche aléatoire branchante ; mouvement brownien branchant ; excursion brownienne ; sélection.

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Abstract

We consider branching Brownian motion which is a mathematical object modeling the evolution of a population. In this system, particles diffuse on the real line according to Brownian motions and branch independently into two particles at rate 1. We are interested in the rightmost (resp. leftmost) position at time t, which is defined as the maximum (resp. minimum) among the positions occupied by the particles alive at this time. According to Lalley and Sellke [101], every particle born in this system will have a descendant reaching the rightmost position at some future time. We study this phenomenon quantitatively, by estimating the first time when every particle alive at time s has had such a descendant.

We then study an analogous model the branching random walk in discrete time, in which random walks are indexed by a Galton-Watson tree. Similarly, we define the rightmost and the leftmost positions at the n-th generation. We consider the walk starting from the root which ends at the leftmost position. We show that this work, after being properly rescaled, converges in law to a normalized Brownian excursion.

The last part of the thesis concerns the evolution of a population with selection. Given a regular tree in which each individual has N children, we attach to each individual a random variable. All these variables are i.i.d., uniformly distributed in $[0,1]$. Selection applies as follows. An individual is kept if along the shortest path from the root to the individual, the attached random variables are increasing. All other individuals are killed. We study the asymptotic behaviors of the evolution of the population when N goes to infinity.

Keywords : Branching random walk ; branching Brownian motion ; Brownian excursion ; selection.

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Chapitre 1

Introduction

Cette thèse concerne principalement le modèle de la marche aléatoire branchante (BRW), ainsi que son analogue à temps continu : le mouvement brownien branchant (BBM). Dans la théorie des probabilités, les marches aléatoires branchantes, qui modélisent l'évolution d'une population, généralisent les processus de branchement usuels en ajoutant un élément spatial aux individus. Un processus de sélection peut aussi intervenir, basé sur ces éléments spatiaux.

Ce travail se décompose en quatre chapitres (Chapitres 2-5), formés chacun d'un article indépendant. Le Chapitre 2, basé sur Chen [55], étudie un processus de branchement avec mutations neutres en regroupant les individus selon leurs allèles. Le Chapitre 3 (Chen [57]) basé sur l'étude du mouvement brownien branchant binaire, donne une compréhension quantitative d'un phénomène intéressant (découvert par Lalley et Sellke [101]) qui dit que chaque individu apparu peut avoir à un instant futur un descendant qui occupe la position la plus à droite. Le Chapitre 4 (Chen [56]) porte sur les marches aléatoires branchantes à temps discret, et concerne l'historique des individus atteignant les positions extrémales. Le Chapitre 5, qui se concentre sur un modèle venant de l'évolution en biologie, décrit des comportements d'une population en présence d'une sélection forte.

Nous introduisons maintenant les modèles à temps discret et continu qui sont l'objet d'études de cette thèse, et donnons une brève revue de la littérature sur le sujet, avant d'énoncer les résultats obtenus dans cette thèse.

1.1 Définition des modèles

1.1.1 Processus de branchement

Le premier processus de branchement étudié est le processus de Galton-Watson, qui a été introduit par Sir Francis Galton en 1873, puis étudié par Galton et Watson [70] (voir Kendall [92] pour un récit détaillé). Néanmoins, le travail de Bienaymé [33] en 1845 sur ce sujet a été remarqué par Heyde et Seneta [82] et ce processus s'écrit parfois le processus de Bienaymé-Galton-Watson. Un processus de Galton-Walton $(Z_n)_{n\geq 0}$ s'intéresse à l'évolution d'une population dans laquelle chaque individu produit indépendamment un nombre aléatoire identiquement distribué d'enfants. On définit Z_n le nombre d'individus présents à la génération n pour tout $n \geq 0$. La loi du nombre d'enfants est appelée loi de reproduction, qui permet d'obtenir celle du processus $(Z_n)_{n\geq 0}$.

Étant donné la loi de reproduction du processus $P = \left(p_k\right)_{k\geq 0}$, la distribution de $\left(Z_n\right)_{n\geq 0}$ peut être caractérisée de la manière suivante :

$$Z_0 = 1 \quad \text{ et } \quad Z_{n+1} = \sum_{i=1}^{Z_n} X_{i,n}, \tag{1.1.1}$$

où $\{X_{i,n};\ i,n \geq 0\}$ est une famille de variables aléatoires i.i.d. de loi P. La fonction génératrice de la distribution P est définie par $\psi(s) := \sum_{k\geq 0} p_k s^k = \mathbb{E}[s^{Z_1}]$. Récursivement, $\mathbb{E}[s^{Z_n}] = \psi^{\circ n}(s) = \psi \circ \cdots \circ \psi(s)$. En étudiant la fonction ψ, on comprend des propriétés du processus (Z_n). Par exemple, la probabilité de survie, qui s'exprime par

$$\mathbb{P}(Z_n > 0, \forall n \geq 0) = \lim_{n\to\infty} \mathbb{P}(Z_n > 0), \tag{1.1.2}$$

est la plus petite solution dans l'intervalle $[0,1]$ de l'équation $\psi(s) = s$. Ce résultat nous donne un critère d'extinction de la population : le processus s'éteint presque sûrement si et seulement si $\sum_{k\geq 0} kp_k \leq 1$. En conséquence, on peut classer les processus de Galton-Watson en trois groupes : surcritique, critique ou souscritique, selon que $\sum_{k\geq 0} kp_k > 1$, $= 1$ ou < 1.

Notons aussi que $Z_n/\mathbb{E}[Z_n]$ est une martingale positive, dont la limite peut être non triviale dans le cas surcritique. Cela nous amène au théorème de Kesten-Stigum [94], qui dit que cette limite est non dégénérée si et seulement si $\mathbb{E}[Z_1 \log Z_1] < \infty$. Dans le cas où $\mathbb{E}[Z_1 \log Z_1] = \infty$, on a $Z_n/\mathbb{E}[Z_n] \to 0$ presque sûrement lorsque $n \to \infty$. La bonne renormalisation de Z_n telle qu'une limite non dégénérée existe a été trouvée par Seneta [131] et améliorée par Heyde [81]. On peut consulter les livres d'Athreya et Ney [22] et d'Harris [79] pour les résultats généraux sur le comportement du processus de Galton-Watson.

Dans un contexte à temps continu, on peut construire un processus de branchement $(Z_t)_{t\geq 0}$ dans lequel chaque individu se divise indépendamment à taux $\beta > 0$ (c'est-à-dire que chaque individu survit un temps exponentiel de paramètre β et est remplacé par ses enfants à l'instant de mort) en des enfants dont le nombre suit la loi $P = (p_k)_{k\geq 0}$.

Ajoutant des éléments au modèle nous donne des variations du processus de branchement qui contiennent plus de paramètres. Par exemple, nous pouvons attacher un *type* à chaque individu et faire en sorte que les descendants évoluent en fonction des types de leurs ancêtres. Le processus de branchement est ainsi enrichi. Par exemple, un type peut désigner un allèle et le processus enrichi modélise l'évolution d'une population via mutation d'allèles.

En faisant agir l'élément spatial, le type d'un individu reflète sa déplacement et le système s'étend dans un espace vectoriel ($\mathbb{R}$ par exemple). On crée ainsi le mouvement brownien branchant (BBM) et la marche aléatoire branchante (BRW) sur $\mathbb{R}$. En un sens plus général, le type attaché à un individu peut être un vecteur dans $\mathbb{R}_+ \times \mathbb{R}$ dont la première coordonnée désigne l'âge de son parent à la naissance de cet individu et la seconde désigne son déplacement dans l'espace. On obtient alors un processus de Crump-Mode-Jagers (Biggins [36, 37]) dans lequel les individus peuvent également se déplacer au cours de leur vie. Le BBM devient un exemple particulier de ce processus.

1.1.2 Processus de branchement avec mutations neutres

Dans ce modèle, chaque type correspond à un allèle, et nous nous intéressons d'un nombre infini d'allèles (voir [22] pour un modèle de types finis). Nous nous donnons pour commencer un processus de Galton-Watson partant de plusieurs ancêtres. Supposons qu'un enfant peut être soit un clone, qui est du même type (allèle) que son parent, soit un mutant, qui est, d'un type différent de celui de son parent et de celui de tout le reste de la population. Chaque mutant, possède un allèle (type) distinct, et donne naissance à son tour, à des clones de lui-même et à de nouveaux mutants selon la même loi que celle de son parent. Autrement dit, les mutations sont neutres pour la dynamique de la population. Nous utilisons un vecteur aléatoire $\xi = (\xi^{(\mathrm{c})}, \xi^{(\mathrm{m})})$ à valeurs dans $\mathbb{Z}_+^2$, pour décrire respectivement le nombre d'enfants clonaux et celui d'enfants mutants d'un individu. La loi de reproduction du système entier est donnée par $\xi^{(+)} = \xi^{(\mathrm{c})} + \xi^{(\mathrm{m})}$. Dans le cas où chaque mutation surgit indépendamment avec probabilité fixée $p \in (0,1)$, sachant $\xi^{(+)} = \ell$, $\xi^{(\mathrm{m})}$ est une variable binomiale de paramètre (ℓ, p).

Nous pouvons aussi interpréter ce modèle comme celui d'une population dans un environnement spatial, dans lequel un individu soit occupe la même position que son parent, soit immigre vers une nouvelle position. Ce cadre a été beaucoup étudié dans la littérature (Aldous et Pitman [13], Crump et Gillespie [58], Griffiths et Pakes [72], Bertoin [31], Nerman [121], Taïb [132]), ainsi que ses variations (Aldous et Pitman [14], Liggett-Schinazi-Schweinsberg [104], Schinazi et Schweinsberg [130]).

Dans ce modèle, la population est décomposée en groupes (=sous-familles) d'individus ayant le même allèle. Cela aboutit à une partition allélique du système qui est liée à certains processus de coalescence (Ewens [63], Kingman [97], Basdevant et Goldschmidt [24], Dong-Gnedin-Pitman [60]).

1.1.3 Mouvement brownien branchant

Nous considérons le mouvement brownien branchant sur $\mathbb{R}$, qui évolue selon le mécanisme suivant. On démarre avec une particule à l'origine. Chaque particule se déplace selon un mouvement brownien indépendamment du reste du processus. Les particules meurent à taux β et sont remplacées au même endroit par de nouvelles particules dont le nombre suit la distribution $P = (p_k)_{k \geq 0}$. On s'intéresse au cas surcritique où $\sum_{k=0}^{\infty} k p_k > 1$.

Soit $\mathscr{N}(t)$ la collection des particules qui sont vivantes à l'instant t, et soit $X_u(t)$ la postion à l'instant t de la particule $u \in \mathscr{N}(t)$. La position de la particule la plus à droite (à gauche, resp.) au temps t est notée par $R(t) = \max_{u \in \mathscr{N}(t)} X_u(t)$ ($L(t) = \min_{u \in \mathscr{N}(t)} X_u(t)$, resp.).

Soit $u(x,t) = \mathbb{P}\big(R(t) \geq x\big)$. On voit aisément que cette fonction est une solution de l'équation aux dérivées partielles suivante :

$$\frac{\partial}{\partial t} u = \frac{1}{2} \frac{\partial^2}{\partial x^2} u + F(u), \tag{1.1.3}$$

où $F(u) = \beta(1 - u - \psi(1-u))$. Cette équation, dite, F-KPP, a été introduite par Fisher [68] et Kolmogorov-Petrovskii-Piskounov [98]. Elle est un outil important dans l'étude du comportement asymptotique du BBM. Par exemple, en utilisant les résultats de Kolmogorov et al. [98] sur l'onde progressive des solutions de l'équation F-KPP, on a trouvé que la position la plus à droite est asymptotiquement linéaire :

$$\lim_{t \to \infty} \frac{R(t)}{t} = \sqrt{2\beta m}, \text{ p.s.}, \tag{1.1.4}$$

où $m := \sum_{k \geq 0} k p_k - 1$. McKean [120] a utilisé le calcul stochastique pour estimer l'ordre de grandeur de $R(t) - \sqrt{2\beta m} t$ à l'aide de l'équation F-KPP (1.1.3). Récemment, l'équation F-KPP a aussi été utilisée pour étudier le processus vu de la position extrémale, prouvant ainsi une conjecture de Lalley et Sellke [101] (Arguin-Bovier-Kistler [19, 20, 21], Aïdékon-Berestycki-Brunet-Shi [6], Brunet et Derrida [49]).

Notons que le BBM est un modèle de l'évolution de la population. C'est également intéressant d'ajouter un critère de sélection au modèle en éliminant des individus peu favorisés par l'environnement. Par exemple, on ajoute une barrière linéaire de pente ρ et on tue les particules qui tombent au-dessous de cette barrière. Kesten [93] a montré qu'il y a extinction presque sûrement si $\rho \geq \sqrt{2\beta m}$ et survie avec probabilité strictement positive sinon. Le temps d'extinction a été étudié par Harris et Harris [74], Kesten [93] et Berestycki-Berestycki-Schweinsberg [28]. Nous citons également Berestycki et al. [27, 29] pour le cas presque-critique où $\rho = \sqrt{2\beta m - \varepsilon}$ avec $\varepsilon \downarrow 0$, et Harris et al. [77] pour le BBM dans une bande. Étant donné $\rho \geq \sqrt{2\beta m}$, l'estimation de la taille de la population a été réalisée dans Neveu [122], Aldous [12] et Maillard [114]. De plus, dans le cas critique, une barrière du seconde ordre en $O(t^{1/3})$ telle que le système survive a été décrite par Jaffuel [89] (voir aussi Roberts [127]). La compréhension de ce modèle a été obtenue principalement grâce à la technique dite la décomposition en épine qui consiste à exhiber, à l'aide d'un changement de probabilités, une branche particulièrement importante (épine dorsale) dans le BBM (voir Harris et Roberts [78] par exemple). En plus, cette technique probabiliste a été appliquée pour l'analyse des ondes progressives de l'équation F-KPP (Harris et al. [75], Harris [76], Kyprianou [100]).

Un exemple intéressant avec un critère de sélection relatif, est le N-BBM. On démarre avec N particules qui évoluent indépendamment selon le même mécanisme. Dès que le nombre de particules dépasse N, on ne garde que les N particules les plus à droite et on tue toutes les autres. C'est un modèle lié à l'équation F-KPP avec un bruit ou une coupure. On peut se référer aux travaux récents de Brunet et al. [50, 51], de Bérard et Gouéré [25], et de Maillard [112, 113, 115].

Des variations inhomogènes du BBM sont aussi considérées en supposant que le taux instantané de branchement pour une particule située à x est donné par $\beta(x) \geq 0$, où $\beta(\cdot)$ est une fonction continue intégrable (Lalley et Sellke [102, 103]), ou en supposant que les mouvements browniens réalisés par les particules sont de variance $\sigma_T(t)$ dépendant du temps (Fang et Zeitouni [65], Maillard et Zeitouni [116], Mallein [117], Bovier et Hartung [43]).

1.1.4 Marche aléatoire branchante

Nous définissons une marche aléatoire branchante sur $\mathbb{R}$ de la façon suivante. À l'instant $n = 0$, il y a un individu à l'origine, appelé la racine. À l'instant $n = 1$, cet individu meurt et donne naissance à la première génération d'individus dont les positions sont données par un processus ponctuel $\mathscr{L}$ sur $\mathbb{R}$. À l'instant $n = 2$, chaque individu de la première génération meurt et donne naissance à des enfants dont les positions par rapport à leur parent sont données par une copie indépendante de $\mathscr{L}$. Le processus continue indéfiniment (s'il n'y a pas d'extinction) par itérations successives et chaque individu se reproduit indépendamment des autres. On note $\mathbb{T}$ l'arbre généalogique des individus, $|u|$ la génération d'un individu $u \in \mathbb{T}$ et $V(u)$ sa position. En particulier, $\{V(u); |u| = 1\}$ suit la loi de $\mathscr{L}$. On définit aussi la position la plus à gauche (la position minimale) de la génération n par $I_n := \inf_{|u|=n} V(u)$.

Remarquons que la distribution de la BRW dépend totalement de celle de $\mathscr{L}$. Nous introduisons la transformée de Laplace-Stieltjes du processus ponctuel $\mathscr{L}$:

$$\Phi(t) := \mathbb{E}\Big[\sum_{x\in\mathscr{L}} e^{-tx}\Big], \qquad \forall t \in (-\infty, +\infty). \tag{1.1.5}$$

Nous définissons aussi $\Psi(t) := \log \Phi(t)$, qui est convexe. Pour chaque t tel que $\Psi(t)$ est finie, $W_n(t) = \sum_{|u|=n} e^{-tV(u)-n\Psi(t)}$ est une martingale, appelée la martingale additive.

L'étude des marches aléatoires branchantes est beaucoup poussée via des équations fonctionnelles similaires à l'équation F-KPP ou via des outils probabilistes. Parallèlement à (1.1.4), Biggins [34], Hammersley [73], Kingman [95] ont montré que sous l'hypothèse $\Psi(t) < \infty$ pour certain $t > 0$, conditionnellement à la survie, $\mathbb{P}$-p.s.,

$$\lim_{n\to\infty} \frac{I_n}{n} = v_c, \tag{1.1.6}$$

où

$$v_c := -\inf_{t>0} \frac{\Psi(t)}{t} = -\sup\Big\{a \in \mathbb{R} : \sup_{t\geq 0}\Big(ta - \Psi(t)\Big) < 0\Big\}. \tag{1.1.7}$$

Biggins [35] a donné une condition nécessaire et suffisante pour l'existence d'une limite non dégénérée de la martingale additive $W_n(t)$, qui généralise le théorème de Kesten-Stigum. Ce résultat a été redémontré par Lyons [109] en appliquant la décomposition en épine. C'est une méthode très utile et bien développée (voir Lyons et al. [110] et Biggins et Kyprianou [39] par exemple). En particulier, dans le cas frontier (suivant Biggins et Kyprianou [40]) où $\Psi(1) = \Psi'(1) = 0$, la martingale additive $W_n(1) = \sum_{|u|=n} e^{-V(u)}$ converge presque sûrement vers 0. La normalisation de cette martingale $W_n(1)$ a été obtenue par Biggins et Kyprianou [38], Hu et Shi [87], Aïdékon et Shi [10], puis améliorée par Hu [83].

Le point de vue qui consiste à considérer la limite non dégénérée de la martingale additive comme un point fixe d'une transformation linéaire a été développé à de nombreuses reprises (Durrett et Liggett [62], Liu [105], Biggins et Kyprianou [38, 40], Hu [84], Alsmeyer et Meiners [15, 16]).

Le modèle de la marche aléatoire branchante avec une courbe absorbante a aussi été beaucoup étudié. La probabilité de survie du système avec une barrière linéaire a été obtenue par Biggins et al. [41], Derrida et Simon [59], Gantert et al. [71], Bérard et Gouéré [26]. Dans le cas où le système s'éteint, la vitesse d'extinction sous des conditions supplémentaires a été considérée par Aïdékon et Jaffuel [8]. Également, la taille de la population totale a été étudiée dans les travaux d'Aïdékon [3], d'Addario-Berry et Broutin [1] et d'Aïdékon et al. [7].

Les marches aléatoires branchantes sont étroitement liées avec d'autres objets mathématiques, par exemple avec la marche aléatoire en milieu aléatoire sur un arbre. Dans ce modèle, le marcheur se déplace dans un potentiel donné par une marche aléatoire branchante sur l'axe réel. Beaucoup de résultats fins sur cette marche aléatoire en milieu aléatoire sont obtenus grâce à la compréhension du potentiel (Hu et Shi [85, 86], Aïdékon [5], Faraud et al. [66], Andreoletti et Debs [17, 18]). On signale également les cascades multiplicatives de Mandelbrot, qui interprètent les branchements d'un point de vue plus analytique, voir les travaux de Kahane et Peyriere [91], de Mauldin et Williams [118], et de Waymire et Williams [133].

1.1.5 Percolation accessible sur l'arbre

Ce modèle provient de l'étude de l'évolution en biologie. Nous nous donnons un arbre ou un hypercube qui représente l'espace des génotypes. Le *type* attaché à un individu (représentant un génotype) dans cet espace désigne son adaptabilité aux circonstances, qui est souvent décrite par une variable aléatoire. Par exemple, Kingman [96] a introduit le modèle *House of Cards*, dans lequel il suppose que toutes les adaptabilités sont des variables i.i.d. C'est un modèle sous l'hypothèse nulle selon Franke et al. [69]. D'après un certain critère de sélection, des individus (génotypes) plus favorisés par l'environnement sont conservés et notés accessibles. Nous nous intéressons à l'évolution des individus (génotypes) accessibles.

Dans notre cadre, l'espace des génotypes est donné par un arbre N-aire enraciné et les adaptabilités attachées aux individus sont des variables aléatoires i.i.d., de distribution continue. Un individu (génotype) est noté accessible et conservé si le long de son chemin ancestral, les variables attachées sont croissantes. Tous les autres individus (génotypes) sont inaccessibles. Nowak et Krug [123] ont appelé ce modèle la percolation accessible sur un arbre. Davantage de modèles ont été introduits dans Aita et al. [11] et Franke et al. [69].

1.2 Nos résultats

1.2.1 La partition allélique du processus de branchement avec mutations neutres

Dans la première partie de cette thèse, on suppose toujours que $\xi^{(+)}$ est de moyenne 1 et de variance finie. On démarre avec a ancêtres ayant le même allèle et suppose que chaque mutation surgit indépendamment avec probabilité p. La loi du système est notée par $\mathbb{P}_a^p$. Le système s'éteint p.s.

La partition allélique est établie de la manière suivante. Nous ajoutons des marques sur les arrêtes entre les individus et leurs enfants mutants. Il est alors commode de dire qu'un individu est de type k-ième si son chemin ancestral contient exactement k marques. On note T_k le nombre d'individus de type k-ième et M_k le nombre de mutants de type k-ième, ajoutant que les mutants de type 0-ième sont les ancêtres, autrement dit, $\mathbb{P}_a^p(M_0 = a) = 1$.

En regroupant les individus selon leurs allèles, nous construisons l'arbre d'allèles : $\mathscr{A} = \{\mathscr{A}_u; u \in \mathbb{U}\}$ où $\mathbb{U} = \cup_{k \in \mathbb{Z}_+} \mathbb{N}^k \cup \{\varnothing\}$. Nous définissons $\mathscr{A}_\varnothing := T_0$, qui est la taille de la sous-famille sans mutation. Nous énumérons ensuite les M_1 sous-familles (générées respectivement par les mutants du premier type) dans l'ordre décroissant de leurs tailles, avec la convention que les sous-familles de la même taille sont rangées uniformément au hasard. Nous notons $\mathscr{A}_j$ la taille de la j-ième sous-famille du premier type, en ajoutant que $\mathscr{A}_j = 0$ pour $j > M_1$. Nous complétons alors la construction par itération. Nous aussi définissons le degré externe d'un individu $u \in \mathbb{U}$ dans l'arbre d'allèles par $d_u := \max\{j, \mathscr{A}_{uj} > 0\}$.

Les résultats de Bertoin [32] sont les suivants. Sous $\mathbb{P}_a^p$, $\{(T_k, M_{k+1}); k \in \mathbb{Z}_+\}$ est une chaîne de Markov dont la distribution est notée par $\mathscr{L}\left(\{(T_k, M_{k+1}); k \in \mathbb{Z}_+\}, \mathbb{P}_a^p\right)$. Soient $a(n) \sim nx$ et $p(n) \sim cn^{-1}$ où c, x sont deux constantes strictement positives. Soit σ^2 la variance de $\xi^{(+)}$. Lorsque n tend vers l'infini,

$$\mathscr{L}\left(\{(n^{-2}T_k, n^{-1}M_{k+1}); k \in \mathbb{Z}_+\}, \mathbb{P}_{a(n)}^{p(n)}\right) \Rightarrow \{(Z_{k+1}, cZ_{k+1}); k \in \mathbb{Z}_+\}, \tag{1.2.1}$$

où $(Z_k; k \in \mathbb{Z}_+)$ est un processus de branchement à l'espace d'état continu et temps discret (CSBP, voir Jiřina [90]) avec $Z_0 = x/c$, de la mesure de reproduction

$$\nu(dy) = \frac{c}{\sqrt{2\pi\sigma^2 y^3}} \exp\left(-\frac{c^2 y}{2\sigma^2}\right) \mathbf{1}_{(y>0)} dy.$$

De plus, au sens des marginales finies-dimensionnelles,

$$\mathscr{L}\left(\left\{\left(n^{-2}\mathscr{A}_u, n^{-1}d_u\right); u \in \mathbb{U}\right\}, \mathbb{P}_{a(n)}^{p(n)}\right) \Rightarrow \left\{(\mathscr{Z}_u, c\mathscr{Z}_u); u \in \mathbb{U}\right\},$$

où $(\mathscr{Z}_u; u \in \mathbb{U})$ est un CSBP indexé par l'arbre de la mesure de reproduction ν dont la population initiale suit la loi suivante :

$$\frac{\mathbb{P}(\mathscr{Z}_\emptyset \in dy)}{dy} = \frac{x}{\sqrt{2\pi\sigma^2 y^3}} \exp\left\{-\frac{(cy-x)^2}{2\sigma^2 y}\right\} \mathbf{1}_{(y>0)}.$$

Nous observons que $(\frac{c\mathscr{A}_u}{n^2} - \frac{d_u}{n}) \Longrightarrow 0$ pour chaque $u \in \mathbb{U}$. Nous considérons le taux de cette convergence et obtenons le résultat suivant :

Théorème 1.2.1. *Soient $p(n) = \frac{c}{n} + o(\frac{1}{n\sqrt{n}})$ et $a(n) \sim nx$. Alors lorsque $n \to \infty$,*

$$\mathscr{L}\left(\left\{\left(\frac{T_k}{n^2}, \sqrt{n}(\frac{cT_k}{n^2} - \frac{M_{k+1}}{n})\right); k \in \mathbb{Z}_+\right\}, \mathbb{P}^{p(n)}_{a(n)}\right) \Longrightarrow \left\{\left(Z_{k+1}, \mathscr{N}^{(k+1)}_{cZ_{k+1}}\right); k \in \mathbb{Z}_+\right\},$$

où $\{(\mathscr{N}^{(k)}_t, t \geq 0); k \in \mathbb{N}\}$ est une série de mouvements browniens i.i.d. qui est indépendante de $\{Z_k; k \in \mathbb{Z}_+\}$.

De plus, soit $\{(\gamma^{(u)}_t, t \geq 0); u \in \mathbb{U}\}$ une collection de mouvements browniens i.i.d. qui est indépendante de toutes les autres variables aléatoires, on a

$$\mathscr{L}\left(\left\{\left(\frac{\mathscr{A}_u}{n^2}, \sqrt{n}(\frac{c\mathscr{A}_u}{n^2} - \frac{d_u}{n})\right); u \in \mathbb{U}\right\}, \mathbb{P}^{p(n)}_{a(n)}\right) \Longrightarrow \left\{\left(\mathscr{Z}_u, \gamma^{(u)}_{c\mathscr{Z}_u}\right); u \in \mathbb{U}\right\},$$

au sens des marginales finies-dimensionnelles.

En effet, grâce à la reconstruction de la loi de l'arbre d'allèles par une marche aléatoire sur $\mathbb{R}$ (voir par exemple la section 6.2 dans Pitman [124]), nous établissons ce théorème central limite en utilisant le principe d'invariance de Donsker.

1.2.2 Les temps d'atteinte de la position la plus à droite dans le BBM

La deuxième partie de ce travail concerne le mouvement brownien branchant binaire avec $\beta = 1$. Notons le processus par $\mathbb{X}(s) := \{X_u(s), u \in \mathscr{N}(s)\}$, $\forall s \geq 0$. Les positions extrémales sont données par

$$R(s) = \max_{u \in \mathscr{N}(s)} X_u(s), \quad L(s) = \min_{u \in \mathscr{N}(s)} X_u(s).$$

Reppelons que presque sûrement, $R(t)/t \to \sqrt{2}$. Ensuite, l'étude aprofondie de Bramson [44, 45], en utilisant beaucoup de techniques probabilistes, a démontré la convergence en loi de $R(t) - m(t)$ où $m(t) := \sqrt{2}t - \frac{3}{2\sqrt{2}} \log t$. La limite de cette convergence, d'après Lalley et Sellke [101], peut être représentée par un décalage aléatoire de la distribution de Gumbel. Les fluctuations suivantes ont été obtenues initialement par Hu et Shi [87] dans un context à temps discret et été reprouvées pour le BBM par Roberts [128] :

$$\limsup_{t\to\infty} \frac{R(t) - \sqrt{2}t}{\sqrt{2}\log t} = -\frac{1}{2}, \text{ et } \liminf_{t\to\infty} \frac{R(t) - \sqrt{2}t}{\sqrt{2}\log t} = -\frac{3}{2}, \mathbb{P}\text{-p.s.} \tag{1.2.2}$$

Nous nous intéressons à un phénomène qui a été mis en avant par Lalley et Sellke [101]. Ils ont démontré que chaque particule née dans ce processus aura, dans l'avenir, un descendant qui occupe la position la plus à droite. Autrement dit, la position la plus à gauche $R(\cdot)$ n'est pas occupée exclusivement par les descendants d'un petit nombre de familles au cours du temps.

Nous donnons une description quantitative de ce phénomène. Pour chaque particule $u \in \mathscr{N}(s)$ vivante à l'instant s, le temps d'attente tel que cette particule u a un descendant atteignant la position la plus à droite est

$$\tau_u := \inf\left\{t > 0 : R(t+s) = \max_{\substack{u \le v,\\ v \in \mathscr{N}(t+s)}} X_v(t+s)\right\},$$

où $u \le v$ désigne que v est u elle-même ou un descendant de u.

Notons $\ell(s) \in \mathscr{N}(s)$ la particule située la plus à gauche à l'instant s.

Théorème 1.2.2. *$\mathbb{P}$-presque sûrement, on a*

$$\lim_{s\to\infty} \frac{\log \tau_{\ell(s)}}{s} = 4. \tag{1.2.3}$$

Cependant, la particule la plus à gauche n'est pas celle qui "traîne les pieds" dans l'ensemble $\mathscr{N}(s)$. Compte tenu des positions de toutes les particules vivantes au temps s, ainsi que leurs évolutions, nous obtenons le résultat principal suivant.

Théorème 1.2.3. *Soit $\Theta_s := \max_{u\in\mathscr{N}(s)} \tau_u$. $\mathbb{P}$-presque sûrement, on a*

$$\lim_{s\to\infty} \frac{\log \Theta_s}{s} = 2+2\sqrt{2} > 4. \tag{1.2.4}$$

En effet, la preuve du théorème révèle que le plus grand τ_u pour $u \in \mathscr{N}(s)$ est atteint par une particule située à une position autour de $-(2-\sqrt{2})s$ qui survit longtemps et qui se déplace vers la gauche aussi loin que possible. Un tel résultat est obtenu parce que le temps d'attente d'une particule est influé par des facteurs multiples. Le point de départ $X_u(s)$ pour $u \in \mathscr{N}(s)$ en est un, par exemple. En plus, un individu $u \in \mathscr{N}(s)$ qui produit immédiatement beaucoup de descendants pourrais avoir un descendant "extraordinaire" facilement. Cet événement favorise la chance de réduire le temps d'attente τ_u. Ainsi, si un individu part d'une position à gauche, produit peu d'enfants et se déplace vers la gauche, son temps d'attente serais beaucoup plus long que les autres.

C'est assez naturel de poser des questions similaires pour les marches aléatoires branchantes. Dans ce cas-là, nous supposons des conditions de moments pour les déplacements. Soit $P =$

$(p_k)_{k\geq 0}$ la loi de nombre d'enfants. Si $p_0 = p_1 = 0$, la population explose rapidement ; c'est la particule située la plus à gauche qui nous fournit le plus grand temps d'attente. Si $p_0 = 0 < p_1$, nous obtenons des résultats similaires à ceux du BBM.

1.2.3 La marche allant à la position la plus à gauche dans la BRW

Dans la troisième partie, nous travaillons sur la marche aléatoire branchante. Nous nous concentrons sur le cas où $\Psi(1) = \Psi'(1) = 0$ et $\Psi(0) > 0$. Sous cette hypothèse, la probabilité de survie est strictement positive. Tel que nous avons indiqué au-dessus, la martingale additive $W_n := \sum_{|u|=n} e^{-V(u)}$ converge presque sûrement vers 0 lorsque $n \to \infty$. En conséquence, $I_n \to \infty$ p.s. Rappelons aussi que $I_n/n \to 0$ p.s. ($v_c = 0$ dans (1.1.7)).

Le seconde ordre de I_n a été étudié par McDiarmid [119], puis a été trouvé séparément par Hu et Shi [87] et par Addario-Berry et Reed [2], et est prouvé égal à $\frac{3}{2}\log n$ en probabilité. Notons qu'il existe des fluctuations p.s. ([87]). Dans [2], les auteurs ont estimé l'espérance de I_n et ont démontré, sous des hypothèses convenables, la tension de I_n autour de sa moyenne. Bramson et Zeitouni [46], via des équations récursives, ont obtenu la tension de I_n autour de sa médiane $\frac{3}{2}\log n + O(1)$, sous des hypothèses sur la queue distribution de $\mathscr{L}$. Dans un cas très particulier où les déplacements sont i.i.d. et admettent une densité log-concave, Bachmann [23] a prouvé la convergence en loi de I_n autour de sa médiane. Puis, Aïdékon [5], en exclusant le cas où la loi de $\mathscr{L}$ est supportée par un sous-groupe discret de $\mathbb{R}$ (lattice), a généralisé ce résultat et a prouvé que $I_n - \frac{3}{2}\log n$ converge en loi vers un décalage aléatoire de la distribution de Gumbel, comme un analogue à celui de Bramson [45] pour le BBM. Par la suite, Madaule [111] a étudié le processus à temps discret vu de la position extrémale.

Nous considérons la branche partant de la racine qui va à la position la plus à gauche à l'instant n. La racine de l'arbre $\mathbb{T}$ est notée par e. Pour chaque noeud $u \in \mathbb{T} \setminus \{e\}$, nous écrivons $[\![e,u]\!] = \{e, u_1, \dots, u_{|u|} = u\}$ pour le chemin le plus court connectant la racine e et u (chemin ancestral de u), avec $|u_k| = k$ pour tout $1 \leq k \leq |u|$. Ainsi, u_k est l'ancêtre de u à la k-ième génération.

Si $I_n < \infty$, c'est-à-dire, si la marche aléatoire branchante survit jusqu'à la génération n, on note $m_n^{(n)}$ un sommet choisi uniformément dans l'ensemble $\{u : |u| = n, V(u) = I_n\}$ des particules qui réalisent la position la plus à gauche à l'instant n. Soit $[\![e, m_n^{(n)}]\!] = \{e =: m_0^{(n)}, m_1^{(n)}, \dots, m_n^{(n)}\}$ le chemin ancestral de $m_n^{(n)}$. Nous considérons la trajectoire suivie par $m_n^{(n)}$:

$$(I_n(k); 0 \leq k \leq n) := (V(m_k^{(n)}); 0 \leq k \leq n). \tag{1.2.5}$$

Évidemment, $I_n(0) = 0$ et $I_n(n) = I_n$. En utilisant la décomposition en épine à l'aide de la martingale W_n, nous arrivons au résultat principal suivant.

Théorème 1.2.4. *Supposons que*

$$\sigma^2 := \mathbb{E}\Big[\sum_{|u|=1} V(u)^2 e^{-V(u)}\Big] \in (0, \infty), \tag{1.2.6}$$

et que

$$\mathbb{E}[X(\log_+ X)^2] < \infty,\ \mathbb{E}[\widetilde{X} \log_+ \widetilde{X}] < \infty,\ ^1 \tag{1.2.7}$$

1. $\log_+ y := \log\Big(\max(y,1)\Big),\ \forall y \in \mathbb{R},$

où $X := \sum_{|u|=1} e^{-V(u)}$ *et* $\widetilde{X} := \sum_{|u|=1} V(u)_+ e^{-V(u)}$.[2]

La trajectoire renormalisée $(\frac{I_n(\lfloor sn \rfloor)}{\sqrt{\sigma^2 n}}; 0 \leq s \leq 1)$ *converge en loi vers une excursion brownienne renormalisée* $(e_s; 0 \leq s \leq 1)$.

Ce résultat implique que la trajectoire suivie par la particule $m_n^{(n)}$, avant de terminer en bas au bout de n étapes, monte et reste longtemps à une hauteur de l'ordre $\sqrt{n}$. On rappelle un résultat de Fang et Zeitouni [64] et de Faraud et al. [66] qui dit qu'il existe p.s. une branche restant toujours au-dessous de $b_c n^{1/3}$ jusqu'à l'instant n, où $b_c := \left(3\pi^2\sigma^2/2\right)^{1/3}$. Signalons aussi que dans [6], pour le modèle du BBM, les auteurs ont prouvé que la trajectoire à temps reversé suivie par la particule la plus à gauche converge en loi vers un certain processus stochastique non dégénéré.

Notre approche est inspirée par celle d'Aïdékon [4] dans lequel la convergence en loi de $I_n - \frac{3}{2}\log n$ est obtenue en supposant que $\mathscr{L}$ est non-lattice. Dans ce travail, la conclusion est valable à la fois pour $\mathscr{L}$ lattice et non-lattice.

1.2.4 La population des individus accessibles via un chemin croissant

Dans la dernière partie de cette thèse, on étudie un arbre N-aire, $T^{(N)}$, dont la racine est $\varnothing$. Pour chaque sommet $\sigma \in T^{(N)}$, nous attachons une variable aléatoire continue, notée x_σ. Supposons que toutes ces x_σ, $\sigma \in T^{(N)}$ sont i.i.d. Comme précédemment, on note $|\sigma|$ la génération de σ et σ_i (pour $0 \leq i \leq |\sigma|$) son ancêtre à la génération i. Le chemin ancestral de σ est $[\![\varnothing, \sigma]\!] := \{\sigma_0 := \varnothing, \sigma_1, \cdots, \sigma_{|\sigma|} := \sigma\}$. Un sommet σ est dit accessible et gardé dans l'arbre si le long de son chemin

2. $y_+ := \max(y, 0)$, $\forall y \in \mathbb{R}$.

ancestral, les variables attachées sont croissantes, autrement dit,

$$\sigma \text{ accessible } \Leftrightarrow x_\varnothing < x_{\sigma_1} < \cdots < x_\sigma. \tag{1.2.8}$$

On enlève tous les autres sommets. On dit aussi que le chemin $[\![\varnothing, \sigma]\!]$ est accessible si σ est accessible. Pour tout $k \geq 1$, soit $\mathscr{A}_{N,k} := \{\sigma \in T^{(N)} : |\sigma| = k, \sigma \text{ accessible}\}$. Nous définissons :

$$Z_{N,k} := \sum_{|\sigma|=k} 1_{(\sigma \in \mathscr{A}_{N,k})}.$$

La distribution explicite de x_σ ne change pas celle de $Z_{N,k}$. On suppose que les variables attachées suivent la loi uniforme sur l'intervalle $[0,1]$. Nowak et Krug [123] ont étudié $\mathbb{P}[Z_{N,m(N)} \geq 1]$ où $m(N)$ dépend de N. Berestycki, Brunet et Shi [30] et Hegarty et Martinsson [80] ont considéré un modèle similaire mais sur un hypercube N-dimensionnel.

Nous notons l'entier $\lfloor \alpha N \rfloor$ par αN dans cette partie, et nous nous intéressons au comportement de $Z_{N,\alpha N}$ avec $\alpha > 0$ et N grand. Pour tout $x \in [0,1]$, nous définissons $\mathbb{P}_x(\cdot) := \mathbb{P}(\cdot | x_\varnothing = x)$. Les résultats obtenus sont les suivants.

Théorème 1.2.5. *Soit* $\theta(\alpha) := \alpha(1 - \log \alpha)$ *pour* $\alpha > 0$.

(i) Si $\alpha \in (0, e)$, $\mathbb{P}_0$*-presque sûrement,*

$$\lim_{N \to \infty} \frac{Z_{N,\alpha N}}{N} = \theta(\alpha) > 0. \tag{1.2.9}$$

(ii) Si $\alpha = e$,

$$\mathbb{P}_0\Big[Z_{N,\alpha N} \geq 1\Big] = N^{-3/2+o_N(1)} \text{ lorsque } N \to \infty, \tag{1.2.10}$$

où $o_N(1)$ est une série de réels qui tend vers zéro.

(iii) Si $\alpha > e$,

$$\lim_{N\to\infty} \frac{\log \mathbb{P}_0\Big[Z_{N,\alpha N} \geq 1\Big]}{N} = \theta(\alpha) < 0. \tag{1.2.11}$$

On observe donc que la population accessible augmente d'abord rapidement, et que'elle diminue à partir de la génération N. Vers la génération critique eN, la population s'éteint rapidement. En particulier, on obtient une convergence en loi pour $\alpha = 1$ (Théorème 5.1.4 du Chapitre 5).

1.2.5 Perspectives

Dans l'étude des marches aléatoires branchantes dans le cas où $\Psi(1) = \Psi'(1) = 0$ et $\Psi(0) > 0$, la martingale dérivée $D_n = \sum_{|u|=n} V(u)e^{-V(u)}$ a été introduite. Sous l'hypothèse (1.2.6), cette martingale converge p.s. vers une limite $D_\infty \in [0,\infty)$ qui satisifait l'équation suivante (cascade de Mandelbrot, voir Liu [106, 107] par exemple) :

$$D_\infty \stackrel{d}{=} \sum_{|u|=1} e^{-V(u)} D_\infty^{(u)}, \tag{1.2.12}$$

où, conditionnellement à $\{V(u); |u| = 1\}$, $D_\infty^{(u)}$ sont des copies indépendantes de D_∞ (Théorème 5.1 dans [39]). Je m'intéresse à une condition nécessaire et suffisante pour que cette limite D_∞ ne soit pas triviale. Pour cela, j'introduis une martingale positive en ajoutant une barrière à la BRW :

$D_n^{(\alpha)} := \sum_{|u|=n} R(V(u)+\alpha)e^{-V(u)}\mathbf{1}_{(V(u_k)\geq -\alpha, \forall k\leq n)}$, où $\alpha \geq 0$ et $R(\cdot)$ est la fonction de renouvellement d'une marche aléatoire centrée et de variance finie sur $\mathbb{R}$. Notons que sur $\{\inf_{u\in\mathbb{T}} V(u) \geq -\alpha\}$, $\{D_\infty > 0\} = \{D_\infty^{(\alpha)} := \lim_{n\to\infty} D_n^{(\alpha)} > 0\}$ p.s. Biggins et Kyprianou [39] ont étudié la condition optimale pour la trivialité de $D_\infty^{(\alpha)}$ et celle de D_∞ en utilisant une décomposition en épine via la martingale $D_n^{(\alpha)}$. Aïdékon [4] a prouvé que la condition (1.2.7) est suffisante pour que $D_n^{(\alpha)}$ converge en L^1, donc est suffisante pour que D_∞ soit non dégénérée. Je pense que (1.2.7) serait aussi une condition nécessaire et voudrais obtenir un résultat analogue à celui de Ren et Yang [125] pour le BBM. Je m'intéresse également au lien entre le comportement de D_∞ et l'évolution du processus $(V(u), u\in\mathbb{T})$.

J'ai considéré l'historique de l'individu situé à la position la plus à gauche dans la BRW dans cette thèse. Je m'intéresse aussi à l'historique d'autres individus. Par exemple, je veux considérer les variables $\overline{V}(u) := \max_{v\in[\![e,u]\!]} V(v)$ et estimer les nombres $\sum_{|u|=n} 1_{(\overline{V}(u)\leq n^\gamma)}$ où $\gamma \in (1/3, 1)$. Considérons une marche aléatoire en milieu aléatoire (RWRE) $(X_n)_{n\geq 0}$ sur un arbre qui part de la racine et évolue dans un potentiel donné par la BRW $(V(u), u\in\mathbb{T})$. Sachant l'environnement (le potentiel), la probabilité que X_n a visité un noeud $u\in\mathbb{T}$ avant de retourner à la racine est liée à la valeur $\sum_{k=1}^{|u|} e^{V(u_k)}$, donc à $|u|e^{\overline{V}(u)}$. À partir de cette idée, Faraud et al. [66] ont démontré la convergence p.s. de $\max_{k\leq n} |X_n|/(\log n)^3$ dans le cas où $\min_{t\in[0,1]} \Psi(t) = 0$ et $\Psi'(1) \geq 0$, en étudiant le comportement asymptotique de $\min_{|u|=n} \overline{V}(u)$. Je pense que l'étude des trajectoires dans la BRW pourrait aider à comprendre le comportement de la RWRE X_n.

Je m'intéresse également à la marche aléatoire branchante dans un contexte inhomogène où la loi de reproduction $\mathscr{L}$ change au fil du temps. Le premier et le seconde ordres de la position maximale (la position la plus à droite) ont été étudiés dans [65, 116, 117] dans un cadre particulier. Ce modèle inhomogène est lié à celui de fractales aléatoires établies d'une manière récursive tel que les vecteurs contractants ne sont pas identiquement distribués à chaque étape. La dimension de Hausdorff de ces fractales aléatoires a été démontrée par Liu, Wen et Wu [108] en étudiant des martingales associées. Je veux étudier d'autres propriétés géométriques de ces fractales à l'aide de la BRW inhomogène.

Sur la sélection de l'accessibilité sur un arbre N-aire, la probabilité de survie dans le cas critique où $\alpha = e$ est de l'ordre $N^{-3/2+o_N(1)}$, où $o_N(1) \leq O((\log N)^{-1/2})$. Je veux l'améliorer en précisant ce $o_N(1)$. Dans ce cas critique, je me demande aussi si un théorème de type Yaglom pourrait être prouvé. Je m'intéresse également à la généalogie de la population accessible, qui reste un problème ouvert actuellement. De plus, la loi du système peut être obtenue par une marche aléatoire branchante avec une barrière. Je souhaiterais continuer l'étude à partir de ce point de vue.

Notons que nos résultats dans le Chapitre 5 restent valables en remplaçant l'arbre N-aire $\mathbb{T}^N$ par un arbre de Galton-Watson $\mathscr{T}^N$ dont la loi de reproduction est poissonienne de paramètre N. Je m'intéresse aussi à l'accessibilité sur d'autres arbres de Galton-Watson ou sur des graphes plus généraux.

Chapitre 2

Convergence rate of the limit theorem of a Galton-Watson tree with neutral mutations

The results in this chapter are contained in [55].

Abstract. We consider a Galton-Watson branching process with neutral mutations (infinite alleles model), and we decompose the entire population into sub-families of individuals carrying the same allele. Bertoin [32] describes the asymptotic shape of the process of the sizes of the allelic sub-families when the initial population is large and the mutation rate small. The limit in law is a certain continuous state-space branching process (CSBP). In the present work, we obtain a Central Limit Theorem, thus completing Bertoin's work.

Keywords. Branching process ; Lévy-Itô decomposition ; Donsker's invariance principle ; Skorohod's representation.

2.1 Introduction

We consider a Galton-Watson process (see [22]), that is, a population model with asexual reproduction such that at every generation, each individual gives birth to a random number of children according to a fixed offspring distribution and independently of the other individuals in the population. In this paper, we are interested in the situation where a child can be either a clone, that is, of the same genetic type as its parent, or a mutant, that is, of a new genetic type different from its parent. We stress that each mutant has a distinct allele and in turn gives birth to clones of itself and to new mutants according to the same statistical law as its parent, even though it bears a different allele. In other words, we are working with an infinite alleles model where mutations are neutral for the population dynamics.

To simplify the model, we decompose the entire population into clusters (:sub-families) of individuals having the same allele. This partition will be referred to as the *allelic partition*. Its statistics have been studied in the paper [31]. However, our main purpose here is to investigate asymptotical behaviors in law when the size of the population is large (typically as the number of ancestors is large) and mutations are rare. As shown in [32], under some conditions, a non-degenerate limit exists and is conveniently described in terms of a certain continuous state-space branching process in discrete time (:CSBP [90]).

Let us show a rough idea of the above. We consider a fixed reproduction law which is critical and has finite variance, and assume that the Galton-Watson process starts from n ancestors with the same genetic type. We also suppose that neutral mutations affect each child with probability $1/n$. Recall that such a Galton-Watson process becomes extinct after roughly n generations, and that the total population is of order n^2 . So there are only a few mutations at each generation and thus about n different alleles ; furthermore the largest allelic sub-family is of order n^2 and the allelic type of mutants from this sub-family(:outer degree) is of order n. It is natural to consider the asymptotic features of the rescaled size of the allelic partition.

We use the universal tree $\mathbb{U}$, which is the set of finite sequences of integers(with $\emptyset$ as the root) to record the genealogy of alleles, and define the tree of alleles as a random process $(\mathscr{A}, d)$ on $\mathbb{U}$, such that each allele represents a vertex of $\mathbb{U}$ and that the values at vertices are given by the sizes of the corresponding allelic sub-families and the outer degrees, with the convention that the sizes are ranked in decreasing order for each sibling.

When the size of ancestors is of order n and the rate of mutations is of order $1/n$, we denote by $(\mathscr{A}^{(n)}, d^{(n)})$ the corresponding tree of alleles. Then Bertoin's result [32] is that as n goes to infinity, $n^{-2}\mathscr{A}^{(n)}$ and $n^{-1}d^{(n)}$ converge in law towards the same limit (removing a constant factor). The limit describes the genealogy of a CSBP in discrete time, whose law only depends on the variance of the offspring distribution of the Galton-Watson process. This led us to consider $\sqrt{n}\left(n^{-2}\mathscr{A}^{(n)} - n^{-1}d^{(n)}\right)$, which we prove converges in law to a "normal" distribution with mean zero, whose variance is given by the CSBP.

The plan of this paper is as follows. In section 2.2, we present precisely the model and our limit theorems. In section 2.3, we construct the probability structure of the tree of alleles from the random walk. In section 2.4, we prove our central limit improvement of the limit theorems based on the constructions of Section 2.3.

2.2 The model of the tree of alleles

Our basic data is provided by a pair of non-negative integer-valued random variables

$$\xi = (\xi^{(\mathrm{c})}, \xi^{(\mathrm{m})}),$$

which describes the number of clone-children and the number of mutant-children of a typical individual. We are interested in a special situation where mutations affect each child according to a fixed probability p and independently of the other children (in other words, the conditional distribution of $\xi^{(\mathrm{m})}$ given $\xi^{(\mathrm{c})} + \xi^{(\mathrm{m})} = \ell$ is binomial with parameter (ℓ, p)). We define :

$$\xi^{(+)} = \xi^{(\mathrm{c})} + \xi^{(\mathrm{m})},$$

whose law is noted by $\pi^{(+)}$. We assume that

$$\mathbb{E}[\xi^{(+)}] = 1 \text{ and } \mathrm{Var}(\xi^{(+)}) = \sigma^2 \in (0, \infty).$$

We further implicitly exclude the degenerate cases when $\xi^{(\mathrm{c})} = 0$, or $\xi^{(\mathrm{m})} = 0$. For every integer $a \geq 1$, we denote by $\mathbb{P}_a$ the law of a Galton-Watson process with neutral mutations, starting from a ancestors carrying the same genetic type and with reproduction law given by that of $\xi = (\xi^{(\mathrm{c})}, \xi^{(\mathrm{m})})$.

Moreover, we use the notation $\mathbb{P}_a^p$ for the probability measure under which the Galton-Watson process has a ancestors and the mutation rate is p. $\mathscr{L}(\cdot, \mathbb{P}_a^p)$ then refers to the distribution of a random variable or a process under $\mathbb{P}_a^p$.

We now take into account mutations by assigning marks to the edges between parents and their mutant children. Since we are interested in the genealogy of alleles, it is convenient to say that an individual is of the k-th type if its genotype has been affected by k mutations, that is if its ancestral line comprises exactly k marks. We denote by T_k the total population of individuals of the k-th type and by M_k the total number of mutants of k-th type, with the convention that mutants of the 0-th type are the ancestors, i.e. $\mathbb{P}_a(M_0 = a) = 1$. (FIGURE 2.1 gives a sample of such trees.)

In order to describe the genealogy of allelic sub-families as random processes indexed by the universal tree, we introduce the set of finite sequences of positive integers

$$\mathbb{U} := \bigcup_{k \in \mathbb{Z}_+} \mathbb{N}^k,$$

where $\mathbb{N} = \{1, 2, \cdots\}$ and $\mathbb{N}^0 = \{\emptyset\}$. We recall some standard notations in this setting : if $u = (u_1, \cdots, u_k)$ is a vertex at level $k \geq 0$ in $\mathbb{U}$, then the children of u are $uj = (u_1, \cdots, u_k, j)$ for $j \in \mathbb{N}$. We also denote by $|u|$ the level of the vertex u, with the convention that the root has level 0, i.e. $|\emptyset| = 0$.

By taking advantage of the natural tree structure of $\mathbb{U}$, we construct a process $\mathscr{A} = (\mathscr{A}_u; u \in \mathbb{U})$ from the given Galton-Watson process with neutral mutations, to record the genealogy of allelic sub-families together with their sizes. (The allelic partition of the sample tree of FIGURE 2.1 is referred to in FIGURE 2.2.)

First, $\mathscr{A}_\emptyset = T_0$ is the size of the sub-family without mutation. Then recall that M_1 denotes the number of mutants of the first type. We enumerate the M_1 allelic sub-populations of the first type in the decreasing order of their sizes, with the convention that in the case of ties, sub-populations of the same size are ranked uniformly at random. We denote by $\mathscr{A}_j$ the size of j-th allelic sub-population of the first type, agreeing that $\mathscr{A}_j = 0$ if $j > M_1$. We then complete the construction at the next levels by iteration in an obvious way. Specially, if $\mathscr{A}_u = 0$ for some $u \in \mathbb{U}$, then $\mathscr{A}_{uj} = 0$ for all $j \in \mathbb{N}$. Otherwise, we enumerate in the decreasing order of their sizes the allelic sub-populations of type $|u|+1$ which descend from the allelic sub-family indexed by the vertex u, and then $\mathscr{A}_{uj}$ is the size of this j-th sub-family. We call the process $\mathscr{A} = (\mathscr{A}_u; u \in \mathbb{U})$. We define the outer degree of the tree of alleles $\mathscr{A}$ at some vertex $u \in \mathbb{U}$ as

$$d_u := \max\{j \geq 1 : \mathscr{A}_{uj} > 0\},$$

where we agree that $\max \emptyset = 0$. In words, d_u is the number of allelic sub-populations of type $|u|+1$ which descend from the allelic sub-family indexed by the vertex u ; in particular, $d_\emptyset = M_1$.

We observe that

$$T_k = \sum_{|u|=k} \mathscr{A}_u \text{ and } M_{k+1} = \sum_{|u|=k} d_u.$$

We construct a tree-indexed random process in the following definition.

Definition 2.2.1. *Fix $x > 0$ and ν a Lévy measure on $(0, \infty)$ with $\int (1 \wedge y) \nu(dy) < \infty$. A tree-indexed CSBP with reproduction measure ν and initial population of size x is a process $(\mathscr{Z}_u; u \in \mathbb{U})$ with values in $\mathbb{R}_+$ and indexed by the universal tree, whose distribution is characterized by induction on the levels as follows :*

1. *$\mathscr{Z}_\emptyset = x$ a.s. ;*
2. *for every $k \in \mathbb{Z}_+$, conditionally on $(\mathscr{Z}_v; v \in \mathbb{U}, |v| \leq k)$, the sequences $(\mathscr{Z}_{uj})_{j \in \mathbb{N}}$ for the vertices $u \in \mathbb{U}$ at generation $|u| = k$ are independent, and each sequence $(\mathscr{Z}_{uj})_{j \in \mathbb{N}}$ is distributed*

as the family of the atoms of a Poisson random measure on $(0,\infty)$ with intensity $\mathscr{Z}_u\nu$, where atoms are repeated according to their multiplicity, ranked in the decreasing order, and completed by an infinite sequence of 0 if the Poisson measure is finite.

It follows from the definition that $\left(\Sigma_{|u|=k}\mathscr{Z}_u; k\in\mathbb{Z}_+\right)$ is a CSBP in discrete time, with reproduction measure ν and initial population of size x.

We now present the following two statements, which have been obtained by Bertoin [32].

Proposition 2.2.2. *If we consider the regime*

$$a(n)\sim nx \text{ and } p(n)\sim cn^{-1} \text{ where } c,n \text{ are some positive constants,} \tag{2.2.1}$$

then, as $n\to\infty$, the following convergence in law holds :

$$\mathscr{L}\left(\{(n^{-2}T_k, n^{-1}M_{k+1}); k\in\mathbb{Z}_+\}, \mathbb{P}^{p(n)}_{a(n)}\right) \Longrightarrow \{(Z_{k+1}, cZ_{k+1}); k\in\mathbb{Z}_+\}, \tag{2.2.2}$$

where $(Z_k; k\in\mathbb{Z}_+)$ is a CSBP in discrete time with reproduction measure

$$\nu(dy) = \frac{c}{\sqrt{2\pi\sigma^2 y^3}}\exp\left(-\frac{c^2 y}{2\sigma^2}\right)dy, \qquad y>0,$$

and initial population of size x/c.

Theorem 2.2.3. *In the regime (2.2.1), the rescaled tree of alleles $(n^{-2}\mathscr{A}_u, u\in\mathbb{U})$ under $\mathbb{P}^{p(n)}_{a(n)}$ converges in the sense of finite dimensional distributions to the tree-indexed CSBP $(\mathscr{Z}_u; u\in\mathbb{U})$ with reproduction measure ν given in Proposition 2.2.2 and random initial population with inverse Gaussian distribution :*

$$\frac{\mathbb{P}(\mathscr{Z}_\emptyset\in dy)}{dy} = \frac{x}{\sqrt{2\pi\sigma^2 y^3}}\exp\left\{-\frac{(cy-x)^2}{2\sigma^2 y}\right\}\mathbf{1}_{(y>0)}.$$

More precisely, if we also take into account the outer degrees $(d_u; u\in\mathbb{U})$, then the joint convergence in the sense of finite dimensional distributions also holds :

$$\mathscr{L}\left(\left\{\left(n^{-2}\mathscr{A}_u, n^{-1}d_u\right); u\in\mathbb{U}\right\}, \mathbb{P}^{p(n)}_{a(n)}\right) \Longrightarrow \left\{(\mathscr{Z}_u, c\mathscr{Z}_u); u\in\mathbb{U}\right\}.$$

From Theorem 2, it is immediate that $(\frac{c\mathscr{A}_u}{n^2}-\frac{d_u}{n})\Longrightarrow 0$ for any vertex u. Then a natural idea is to study the rate of the convergence, which brings out our main result.

Theorem 2.2.4. *We assume that $p(n) = \frac{c}{n} + o(\frac{1}{n\sqrt{n}})$ and $a(n) \sim nx$, then as $n \to \infty$,*

$$\mathscr{L}\left(\left\{\left(\frac{T_k}{n^2}, \sqrt{n}(\frac{cT_k}{n^2} - \frac{M_{k+1}}{n})\right); k \in \mathbb{Z}_+\right\}, \mathbb{P}^{p(n)}_{a(n)}\right) \Longrightarrow \left\{\left(Z_{k+1}, \mathscr{N}^{(k+1)}_{cZ_{k+1}}\right); k \in \mathbb{Z}_+\right\},$$

where $\{\mathscr{N}^{(k)}_{\cdot}; k \in \mathbb{N}\}$ is a sequence of independent standard Brownian motions which is independent of $\{Z_k; k \in \mathbb{Z}_+\}$.

Furthermore, assuming that $\{\gamma^{(u)}; u \in \mathbb{U}\}$ is a family of i.i.d. standard BM's which is independent of all random variables mentioned above, we have

$$\mathscr{L}\left(\left\{\left(\frac{\mathscr{A}_u}{n^2}, \sqrt{n}(\frac{c\mathscr{A}_u}{n^2} - \frac{d_u}{n})\right); u \in \mathbb{U}\right\}, \mathbb{P}^{p(n)}_{a(n)}\right) \Longrightarrow \left\{\left(\mathscr{Z}_u, \gamma^{(u)}_{c\mathscr{Z}_u}\right); u \in \mathbb{U}\right\},$$

in the sense of finite dimensional distributions. The law of $\{\mathscr{Z}_u; u \in \mathbb{U}\}$ is described in Theorem 2.

Remark 2.2.5. *If we only assume that $p(n) \sim cn^{-1}$ and $a(n) \sim nx$, the convergence in law holds for $\mathscr{L}\left(\left\{\left(\frac{T_k}{n^2}, (\frac{p(n)T_k - M_{k+1}}{\sqrt{n}})\right); k \in \mathbb{Z}_+\right\}, \mathbb{P}^{p(n)}_{a(n)}\right)$. Similarly, the convergence in the sense of finite dimensional distribution holds for $\mathscr{L}\left(\left\{\left(\frac{\mathscr{A}_u}{n^2}, (\frac{p(n)\mathscr{A}_u - d_u}{\sqrt{n}})\right); u \in \mathbb{U}\right\}, \mathbb{P}^{p(n)}_{a(n)}\right)$.*

To prove this theorem, we borrow the key idea from Bertoin [32], by means of the connection between random walks and branching processes.

2.3 The construction from a random walk

We consider a sequence $\{\xi_n = (\xi_n^{(\mathrm{c})}, \xi_n^{(\mathrm{m})}); n \in \mathbb{N}\}$ of i.i.d. random variables distributed as $\xi = (\xi^{(\mathrm{c})}, \xi^{(\mathrm{m})})$, and then the random walk starting from $a \geq 1$ and with steps $\xi^{(\mathrm{c})} - 1$,

$$S_k^{(\mathrm{c})} := a + \xi_1^{(\mathrm{c})} + \cdots + \xi_k^{(\mathrm{c})} - k, \qquad k \in \mathbb{Z}_+.$$

We still use the notation $\mathbb{P}_a$ for the law of $(S_k^{(\mathrm{c})}; k \in \mathbb{Z}_+)$. We define the first hitting times for this random walk

$$\varsigma(j) := \inf\left\{k \in \mathbb{Z}_+; S_k^{(\mathrm{c})} = -j\right\}, \qquad j \in \mathbb{Z}_+.$$

Indeed, the hitting times $\varsigma(\cdot)$ are such that $\varsigma(0) < \varsigma(1) < \cdots$ since the random walk $S^{(\mathrm{c})}$ cannot make negative steps larger than -1. On the other hand, the assumption that $\mathbb{E}[\xi^{(\mathrm{c})}] < 1$ implies that $S_k^{(\mathrm{c})} \to -\infty$ as k goes to infinity. Hence $\varsigma(j) < \infty$ a.s.

Define

$$\Sigma(j) := \sum_{i=1}^{\varsigma(j)} \xi_i^{(\mathrm{m})}.$$

Let $\widetilde{T}_0 := \varsigma(0)$, $\widetilde{M}_1 := \Sigma(0)$ and define for every $k \in \mathbb{N}$ by an implicit recurrence :

$$\begin{aligned}
\widetilde{T}_0 + \cdots + \widetilde{T}_k &= \varsigma(\widetilde{M}_1 + \cdots + \widetilde{M}_k); \\
\widetilde{M}_1 + \cdots + \widetilde{M}_{k+1} &= \Sigma(\widetilde{M}_1 + \cdots + \widetilde{M}_k) = \sum_{i=1}^{\widetilde{T}_0 + \cdots + \widetilde{T}_k} \xi_i^{(\mathrm{m})}.
\end{aligned}$$

It turns out that for every $a \geq 1$, the chains $\{(T_k, M_{k+1}); k \in \mathbb{Z}_+\}$ and $\{(\widetilde{T}_k, \widetilde{M}_{k+1}); k \in \mathbb{Z}_+\}$ have the same distribution under $\mathbb{P}_a$ (See Section 2 of [32].)

More generally, we can apply the sequence (ξ_n) to construct a random process $(\mathscr{A}', d')$ indexed by the universal tree $\mathbb{U}$. To start with, $(\mathscr{A}', d')$ fulfills the following requirements. First, if $\mathscr{A}'_u = 0$ for some $u \in \mathbb{U}$, then $d'_u = 0$ and $\mathscr{A}'_{uj} = 0$ for all $j \in \mathbb{N}$. Second, for every vertex $u \in \mathbb{U}$ such that $\mathscr{A}'_u > 0$,

$$d'_u = \#\{j \in \mathbb{N} : \mathscr{A}'_{uj} > 0\},$$

which is called the outer degree of $\mathscr{A}'$ at u, is a finite number and $\mathscr{A}'_{uj} > 0$ if and only if $j \leq d'_u$. We set $\mathscr{A}'_\emptyset = \varsigma(0)$ and $d'_\emptyset = \Sigma(0)$. Then, define the increments

$$\lambda(j) := \varsigma(j) - \varsigma(j-1) \text{ and } \delta(j) := \Sigma(j) - \Sigma(j-1), \qquad j \geq 1.$$

For vertices at the first level, $\{(\mathscr{A}'_j, d'_j); 1 \leq j \leq d'_\emptyset\}$ is given by the rearrangement of the sequence $\{(\lambda(j), \delta(j));\ 1 \leq j \leq d'_\emptyset\}$ in the decreasing order with respect to the first coordinate $\lambda(j)$ with the usual convention in case of ties. We may then continue with vertices of the next levels by an iteration which should be obvious. (For instance, the random walk corresponding to FIGURE 2.1 is shown in FIGURE 2.3.)

We say that under $\mathbb{P}_a$, the two processes indexed by the universal tree, $(\mathscr{A}_u, d_u)$ and $(\mathscr{A}'_u, d'_u)$ have the same distribution. (See, e.g. Section 2 of [32].)

Based on the construction from a random walk, we build the following strategy to prove our main theorem.

Let $(\xi_k^{(+)}; k \in \mathbb{N})$ be a sequence of i.i.d. copies of $\xi^{(+)}$, then we consider a random walk $(S_k^{(n)}; k \in \mathbb{Z}_+)$ started from $a(n)$ defined by

$$S_k^{(n)} := a(n) + \xi_1^{(+)} + \cdots + \xi_k^{(+)} - k.$$

By Donsker's invariance principle and Skorohod's representation, we may suppose that with probability one

$$\lim_{n\to\infty} n^{-1} S^{(n)}_{\lfloor n^2 t\rfloor} = x + \sigma B_t, \tag{2.3.1}$$

where $(B_t; t \geq 0)$ is a standard Brownian motion and the convergence holds uniformly on every compact time-interval.

For every fixed n, we now decompose each variable $\xi_i^{(+)}$ as the sum $\xi_i^{(+)} = \xi_i^{(\mathrm{c}n)} + \xi_i^{(\mathrm{m}n)}$ by using a Bernoulli sampling ; that is conditionally on $\xi_i^{(+)} = l$, $\xi_i^{(\mathrm{m}n)}$ has the binomial distribution with parameter $(l, p(n))$. We use independent Bernoulli samplings for the different indices i so that the pairs $(\xi_i^{(\mathrm{c}n)}, \xi_i^{(\mathrm{m}n)})$ are i.i.d. and have the same law as ξ under $\mathbb{P}^{p(n)}$. If we define

$$\begin{aligned} S_k^{(\mathrm{m}n)} &:= \xi_1^{(\mathrm{m}n)} + \cdots + \xi_k^{(\mathrm{m}n)}, \\ S_k^{(\mathrm{c}n)} &:= a(n) + \xi_1^{(\mathrm{c}n)} + \cdots + \xi_k^{(\mathrm{c}n)} - k = S_k^{(n)} - S_k^{(\mathrm{m}n)}; \quad \forall k \in \mathbb{Z}_+. \end{aligned}$$

Then, as $n \to \infty$,

$$\left\{ \frac{S^{(\mathrm{m}n)}_{\lfloor n^2 t\rfloor}}{n}, \frac{S^{(\mathrm{c}n)}_{\lfloor n^2 t\rfloor}}{n} \right\} \longrightarrow \Big(ct, x + \sigma B_t - ct \Big), \tag{2.3.2}$$

where the convergence holds a.s., uniformly on every compact time-interval.

We denote by $(\mathscr{G}(k); k \in \mathbb{N})$ the natural filtration generated by the sequence $(\xi_k^{(\mathrm{c}n)}, \xi_k^{(\mathrm{m}n)}; k \in \mathbb{N})$. With a little abuse of notation, we still use ς, Σ, λ, δ, $\tilde{T}$, $\tilde{M}$, $\mathscr{A}'$ and d' to represent, respectively, the corresponding random variables associated with $(\xi_k^{(\mathrm{c}n)}, \xi_k^{(\mathrm{m}n)}; k \in \mathbb{N})$.

For any $y \geq 0$, let $\tau_y := \inf\{t \geq 0 : ct - \sigma B_t > y\}$. It follows immediately from (2.3.2) that $\frac{\varsigma(0)}{n^2}$ converge almost surely to τ_x. Note that $(\tau_y; y \geq 0)$ is a subordinator with no drift and Lévy measure $c^{-1}\nu$ where ν is defined in Proposition 2.2.2.

2.4 The rate of convergence

The construction from the random walk has been used by Bertoin to obtain his theorems in [32], and is still useful to investigate the rate of convergence of $cn^{-2}\mathscr{A} - n^{-1}d \Longrightarrow 0$.

Proof of Theorem 2.2.4 : Let us continue with the settings of the random walks. Still, with probability one, we have the joint convergence (2.3.2). For simplification, we first discuss $cn^{-2}T_0 - n^{-1}M_1$, which is distributed as $c\frac{\varsigma(0)}{n^2} - \frac{S^{(\mathrm{m}n)}_{\varsigma(0)}}{n}$ with $n^{-2}\varsigma(0) \to \tau_x$ a.s.

We write $X_t^{(\mathrm{m}n)} := \sqrt{n}\left(c\frac{\lfloor n^2 t\rfloor}{n^2} - \frac{S^{(\mathrm{m}n)}_{\lfloor n^2 t\rfloor}}{n} \right)$ and $Y_t^{(\mathrm{c}n)} := \frac{S^{(\mathrm{c}n)}_{\lfloor n^2 t\rfloor}}{n}$ for convenience.

Let us admit for the moment the following joint convergence :

$$(X_t^{(\mathrm{m}n)}, Y_t^{(\mathrm{c}n)})_{t\geq 0} \Longrightarrow \{\beta_{ct}, x+\sigma B_t - ct; t\geq 0\}, \tag{2.4.1}$$

where B is the BM given in (2.3.1) and β is another BM independent of B. The proof of (2.4.1) will be presented later.

Recall that $\varsigma(0)$ is a hitting time of the random walk $S^{(\mathrm{c}n)}$. It is sufficient to say that

$$(\sqrt{n}(c\frac{\widetilde{T}_0}{n^2} - \frac{\widetilde{M}_1}{n}), \frac{\widetilde{T}_0}{n^2}) = (X^{(\mathrm{m}n)}_{\varsigma(0)/n^2}, \varsigma(0)/n^2) \Longrightarrow (\beta_{c\tau_x}, \tau_x) \stackrel{d}{=} (\sqrt{c\tau_x}\mathscr{N}_1^{(1)}, \tau_x),$$

where $\stackrel{d}{=}$ means the equivalence of distribution.

Conditionally on $(\widetilde{T}_0, \widetilde{M}_1)$, precisely given $\widetilde{M}_1 = b(n) \sim bn$ with b a positive constant, we consider the random walk with steps $\xi^{(\mathrm{c}n)} - 1$ started from 0 and obtain that

$$\widetilde{T}_1 = \varsigma(b(n)) - \varsigma(0) \text{ and } \widetilde{M}_2 = \sum_{k=\varsigma(0)+1}^{\varsigma(b(n))} \xi_k^{(\mathrm{m}n)},$$

which are independent of $\mathscr{G}(\varsigma(0))$.

(2.4.1) ensures that conditionally on $\widetilde{M}_1 = b(n)$,

$$\begin{aligned}(\sqrt{n}(c\frac{\widetilde{T}_1}{n^2} - \frac{\widetilde{M}_2}{n}), \frac{\widetilde{T}_1}{n^2}) &\Longrightarrow (\beta_{c(\tau_{b+x}-\tau_x)}, \tau_{b+x}-\tau_x) \\ &\stackrel{d}{=} (\sqrt{c(\tau_{b+x}-\tau_x)}\mathscr{N}_1^{(2)}, \tau_{b+x}-\tau_x).\end{aligned}$$

In addition, we observe that $\sqrt{n}(c\frac{\widetilde{T}_1}{n^2} - \frac{\widetilde{M}_2}{n}) = \sum_{k=\varsigma(0)+1}^{\varsigma(b(n))}\left(\frac{c/n - \xi_k^{(\mathrm{m}n)}}{\sqrt{n}}\right)$ is independent of $\mathscr{G}(\varsigma(0))$. It follows that $\mathscr{N}_1^{(1)}$ and $\mathscr{N}_1^{(2)}$ are independent normal variables and that they are both independent of the subordinator τ. Finally, by the construction of $(\widetilde{T}_k, \widetilde{M}_{k+1})$, we get :

$$\mathscr{L}\left(\left\{\left(\frac{T_k}{n^2}, \sqrt{n}(\frac{cT_k}{n^2} - \frac{M_{k+1}}{n})\right); k\in\mathbb{Z}_+\right\}, \mathbb{P}^{p(n)}_{a(n)}\right) \Longrightarrow \left\{Z_{k+1}, \mathscr{N}^{(k+1)}_{cZ_{k+1}}; k\in\mathbb{Z}_+\right\}.$$

This proves the first part of Theorem 2.2.4.

For any $b>0$, we set a Poisson point process on $(0,\infty)$ with intensity $bc^{-1}\nu$. Let $(\alpha_1(b), \alpha_2(b), \cdots)$ stand for the sequence ranked in the decreasing order of the atoms of such a Poisson point process.

Theorem 2.2.3 tells that, conditionally on $d_\emptyset = b(n) \sim bn$, the joint convergence in the sense of finite dimensional distributions can be obtained, that is

$$\mathscr{L}\left\{\left(\left(\frac{\mathscr{A}_u}{n^2}, \frac{d_u}{n}\right); |u|=1\right); \mathbb{P}^{p(n)}_{a(n)}\right\} \Longrightarrow \left\{\left(\alpha_1(b), c\alpha_1(b)\right), \left(\alpha_2(b), c\alpha_2(b)\right), \cdots\right\}.$$

We observe that

$$\sqrt{n}(c\frac{\widetilde{T}_1}{n^2}-\frac{\widetilde{M}_2}{n})=\sum_{j=1}^{d'_\emptyset}\left(c\frac{\mathscr{A}'_j}{n^2}-\frac{d'_j}{n}\right),\quad \frac{\widetilde{T}_1}{n^2}=\sum_{j=1}^{d'_\emptyset}n^{-2}\mathscr{A}'_j.$$

When given $\widetilde{M}_1$ (or $d'_\emptyset$), $\{\mathscr{A}'_j,d'_j;1\le j\le \widetilde{M}_1\}$ is given by the rearrangement of the sequence $\{\lambda(j),\delta(j);1\le j\le\widetilde{M}_1\}$ in the decreasing order with respect to the first coordinate. Meanwhile, $\{\sqrt{n}(c\frac{\lambda(j)}{n^2}-\frac{\delta(j)}{n}),\frac{\lambda(j)}{n^2};1\le j\le\widetilde{M}_1\}$ is actually $\{X^{(\mathrm{m}n)}_{t_j}-X^{(\mathrm{m}n)}_{t_{j-1}},t_j-t_{j-1};t_0\le t_1\le\cdots\le t_{d'_\emptyset}\}$ where $\{t_j=n^{-2}\varsigma(j);1\le j\le d'_\emptyset\}$.

Recall that given $n^{-1}d'_\emptyset=b(n)\sim bn$, then $n^{-2}\varsigma(d'_\emptyset)\to\tau_{b+x}$ a.s.. Theorem 2.2.3 leads to the fact that the rearrangement of the sequence $\{t_j-t_{j-1};1\le j\le d'_\emptyset\}$ in the decreasing order converges in law to the rearrangement of the family of jump sizes $\{(\tau_y-\tau_{y-});x\le y\le x+b\}$ in the decreasing order which can be viewed as $\{\alpha_1(b)\ge\alpha_2(b)\ge\cdots\}$.

Furthermore, the joint convergence (2.4.1) tells us that $\{X^{(\mathrm{m}n)}_{t_j}-X^{(\mathrm{m}n)}_{t_{j-1}},t_j-t_{j-1};t_0\le t_1\le\cdots\le t_{d'_\emptyset}\}$ is asymptotically corresponding to $\{\beta_{\tau_y}-\beta_{\tau_{y-}},\tau_y-\tau_{y-};x\le y\le x+b\}$. The independence between β and B ensures that conditionally on $\{\tau_y-\tau_{y-};x\le y\le x+b\}$, these $\beta_{\tau_y}-\beta_{\tau_{y-}}$ are independent central normal variables with variance $\tau_y-\tau_{y-}$.

Then the rearrangement of the family $\{X^{(\mathrm{m}n)}_{t_j}-X^{(\mathrm{m}n)}_{t_{j-1}},t_j-t_{j-1};t_0\le t_1\le\cdots\le t_{d'_\emptyset}\}$ in the decreasing order of the second coordinate converges in the sense of finite dimensional distributions to $\{\beta^{(k)}_{c\alpha_k(b)},\alpha_k(b);k\ge1\}$ where $(\beta^{(k)},k\ge1)$ is a sequence of i.i.d BM's which is independent of $\{\alpha_1(b),\alpha_2(b),\cdots\}$.

That is to say, under the probability $\mathbb{P}^{p(n)}_{a(n)}$, conditionally on $d'_\emptyset=b(n)\sim bn$,

$$\left\{\sqrt{n}\left(c\frac{\mathscr{A}'_j}{n^2}-\frac{d'_j}{n}\right),\frac{\mathscr{A}'_j}{n^2};1\le j\le d'_\emptyset\right\}\Longrightarrow\left\{\beta^{(j)}_{c\alpha_j(b)},\alpha_j(b);j\ge1\right\}$$

in the sense of finite dimensional distributions.

Since $(\mathscr{A}',d')\stackrel{d}{=}(\mathscr{A},d)$, the general branching property of the tree of alleles (see [32]) can be applied to entail Theorem 2.2.4.

It finally remains to prove (2.4.1). We first check the convergence of finite-dimensional distributions. The independence and stationarity of the increments of $(X^{(\mathrm{m}n)},Y^{(\mathrm{c}n)})$ follow from the independence and identical distribution of $(\xi^{(\mathrm{c}n)}_i,\xi^{(\mathrm{m}n)}_i)_{i\ge1}$. We thus only need to check the convergence for $(X^{(\mathrm{m}n)}_{s_1},Y^{(\mathrm{c}n)}_{s_1})$ with one fixed $s_1\in(0,1]$. Let us compute the Fourier transforms :

$$\Lambda^{(n)}_1\quad:=\quad\mathbb{E}\left[\exp\left(\mathbf{i}\lambda_1X^{(\mathrm{m}n)}_{s_1}+\mathbf{i}\mu_1Y^{(\mathrm{c}n)}_{s_1}\right)\right],$$

where λ_1 and μ_1 are two real constants and $\mathbf{i}:=\sqrt{-1}$.

As $p(n) = \frac{c}{n} + o(\frac{1}{n\sqrt{n}})$ and $a(n) \sim nx$, we have

$$\begin{aligned}\log(\Lambda_1^{(n)}) &= \mathbf{i}\mu_1 \frac{a(n)}{n} - \mathbf{i}\frac{c}{n^2}\sum_{j=1}^{\lfloor n^2 s_1 \rfloor} \mu_1 - \frac{c}{2n^2}\sum_{j=1}^{\lfloor n^2 s_1 \rfloor} \lambda_1^2 - \frac{\sigma^2}{2n^2}\sum_{j=1}^{\lfloor n^2 s_1 \rfloor} \mu_1^2 + o_n(1) \\ &\longrightarrow \mathbf{i}\mu_1 x - \mathbf{i}c\mu_1 s_1 - \frac{c\lambda_1^2 s_1}{2} - \frac{\sigma^2 \mu_1^2 s_1}{2}.\end{aligned}$$

with $o_n(1) \to 0$ as $n \to \infty$. In fact, the weaker assumption that $p(n) \sim cn^{-1}$ cannot ensure this convergence. Hence $\Lambda_1^{(n)} \longrightarrow \mathbb{E}\left[\exp\left(\mathbf{i}\lambda_1 \beta_{cs_1}\right)\right]\mathbb{E}\left[\exp\left(\mathbf{i}\mu_1(x + \sigma B_{s_1} - cs_1)\right)\right]$.

It remains to verify the tightness of $(X_t^{(\mathrm{m}n)}, Y_t^{(\mathrm{c}n)}; 0 \le t \le 1)$. We take $0 \le s \le r \le t \le 1$, then we estimate

$$\Gamma(s,r,t) := \mathbb{E}\left[\left\|(X_t^{(\mathrm{m}n)}, Y_t^{(\mathrm{c}n)}) - (X_r^{(\mathrm{m}n)}, Y_r^{(\mathrm{c}n)})\right\|^2\right]\mathbb{E}\left[\left\|(X_r^{(\mathrm{m}n)}, Y_r^{(\mathrm{c}n)}) - (X_s^{(\mathrm{m}n)}, Y_s^{(\mathrm{c}n)})\right\|^2\right].$$

Let $F(t,r) := \mathbb{E}\left[\left\|(X_t^{(\mathrm{m}n)}, Y_t^{(\mathrm{c}n)}) - (X_r^{(\mathrm{m}n)}, Y_r^{(\mathrm{c}n)})\right\|^2\right]$. Then

$$F(t,r) = \mathbb{E}\left[\frac{1}{n}\Bigg(\sum_{k=\lfloor n^2 r\rfloor+1}^{\lfloor n^2 t\rfloor}(\xi_k^{(\mathrm{m}n)} - \frac{c}{n})\Bigg)^2\right] + \mathbb{E}\left[\frac{1}{n^2}\Bigg(\sum_{k=\lfloor n^2 r\rfloor+1}^{\lfloor n^2 t\rfloor}(\xi_k^{(\mathrm{c}n)} - 1)\Bigg)^2\right].$$

In view of the distribution of $(\xi^{(\mathrm{m}n)}, \xi^{(\mathrm{c}n)})$, we obtain that

$$\begin{aligned}F(t,r) &= \frac{\lfloor n^2 t\rfloor - \lfloor n^2 r\rfloor}{n}\Big(\sigma^2 p(n)^2 + p(n)(1-p(n))\Big) + \left(\frac{\lfloor n^2 t\rfloor - \lfloor n^2 r\rfloor}{\sqrt{n}}(p(n) - c/n)\right)^2 \\ &+ \frac{\lfloor n^2 t\rfloor - \lfloor n^2 r\rfloor}{n^2}\Big(\sigma^2(1-p(n))^2 + p(n)(1-p(n))\Big) + \left(\frac{\lfloor n^2 t\rfloor - \lfloor n^2 r\rfloor}{n}p(n)\right)^2.\end{aligned}$$

If $\lfloor n^2 t\rfloor - \lfloor n^2 s\rfloor \ge 2$, we have $\frac{\lfloor n^2 t\rfloor - \lfloor n^2 r\rfloor}{n^2} \le 2(t-s)$ and $\frac{\lfloor n^2 r\rfloor - \lfloor n^2 s\rfloor}{n^2} \le 2(t-s)$. If $\lfloor n^2 t\rfloor - \lfloor n^2 s\rfloor \le 1$, then $\lfloor n^2 t\rfloor - \lfloor n^2 r\rfloor = 0$ or $\lfloor n^2 r\rfloor - \lfloor n^2 s\rfloor = 0$. Therefore, we can find a constant $C > 0$ such that

$$\Gamma(r,s,t) = F(t,r)F(r,s) \le C[t-s]^2.$$

By application of Theorem 13.5 in [42], we conclude the convergence (2.4.1). $\square$

Note that under the assumption that $p(n) \sim cn^{-1}$ and $a(n) \sim nx$, we can prove that the following conjoint convergence in law holds :

$$(\mathscr{X}_t^{(\mathrm{m}n)}, Y_t^{(\mathrm{c}n)})_{t\ge 0} \Longrightarrow \{\beta_{ct}, x + \sigma B_t - ct; t \ge 0\},$$

where $\mathscr{X}_t^{(mn)} := \left(\frac{p(n)\lfloor n^2 t\rfloor - S^{(mn)}_{\lfloor n^2 t\rfloor}}{\sqrt{n}}\right)$. Following the same arguments as above, we obtain the conclusion in Remark 2.2.5.

Acknowledgements

I would like to thank M. Yor for his advice and help, including introducing Bertoin's paper to me and showing some useful arguments to me.

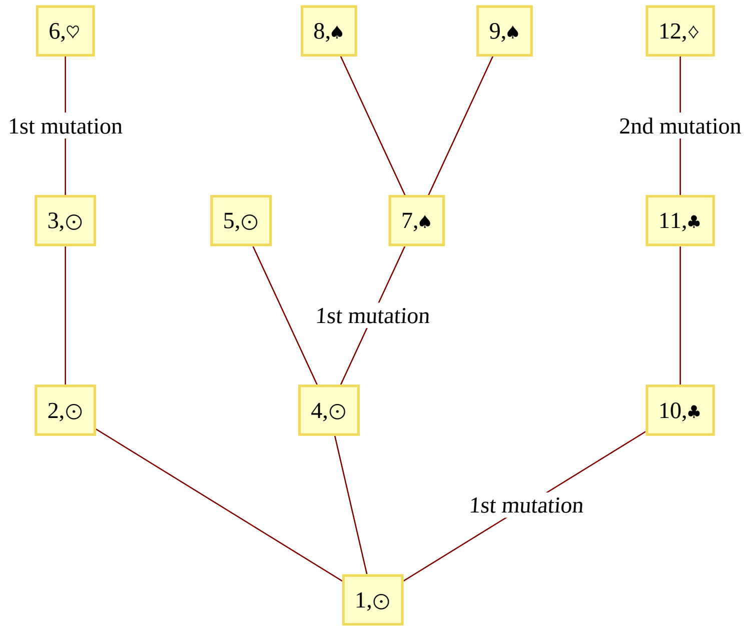

FIGURE 2.1 – The Galton-Watson process with neutral mutations. The symbols $\odot, \heartsuit, \clubsuit, \diamondsuit, \spadesuit$ represent the different alleles. At the same time we enumerate all the vertices.

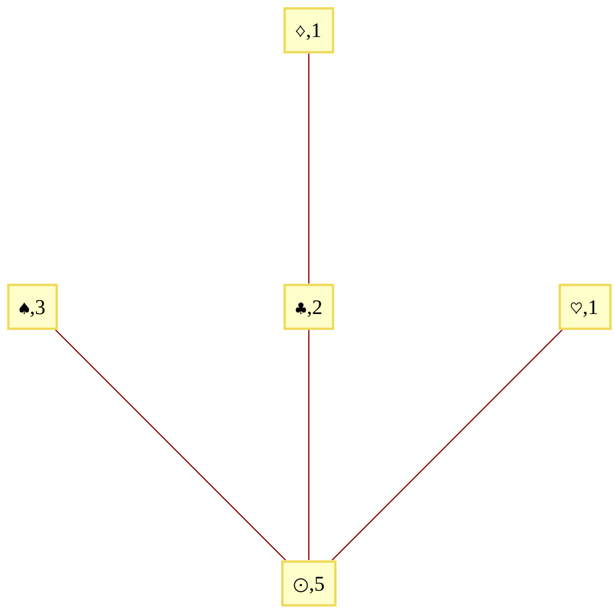

FIGURE 2.2 – The tree of alleles corresponding to the process in FIGURE 2.1. The number of each vertex represents the total size of the cluster of its allele.

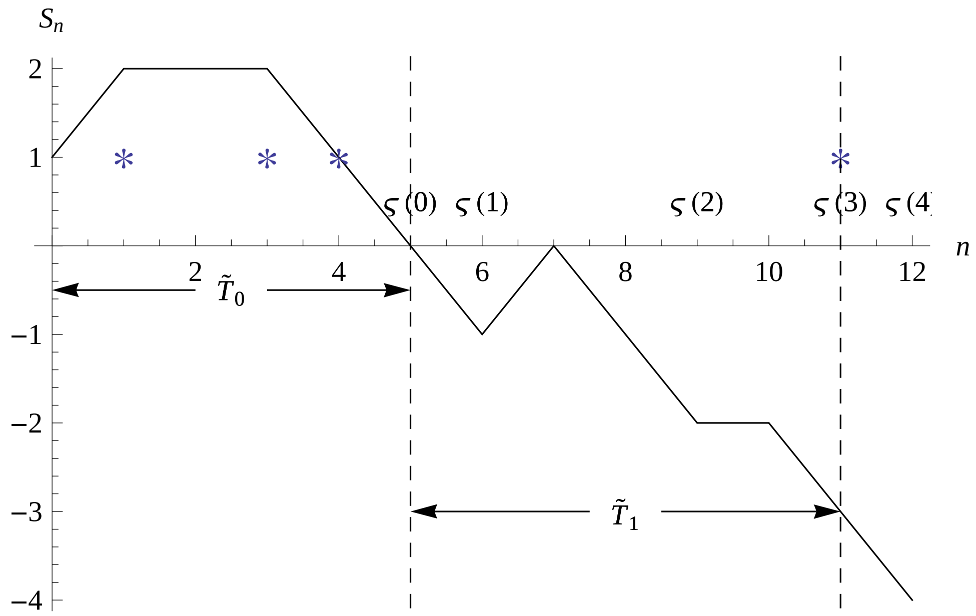

FIGURE 2.3 – The associated random walk with the origin tree shown in FIGURE 2.1.

Chapitre 3

Waiting times for particles in a branching Brownian motion to reach the rightmost position

The results in this chapter are contained in [57].

Summary. It has been proved by Lalley and Sellke [101] that every particle born in a branching Brownian motion has a descendant reaching the rightmost position at some future time. The main goal of the present paper is to estimate asymptotically as s goes to infinity, the first time that every particle alive at the time s has a descendant reaching the rightmost position.

Keywords. Branching Brownian motion, rightmost position.

3.1 Introduction

3.1.1 The model

We consider a branching Brownian motion (BBM) on the real line $\mathbb{R}$, which evolves as follows. Starting at time $t = 0$, one particle located at 0, called the root, moves like a standard Brownian motion until an independent exponentially distributed time with parameter 1. At this time it splits into two particles, which, relative to their birth time and position, behave like independent copies of their parent, thus moving like Brownian motions and branching at rate 1 into two copies of themselves. Let $\mathscr{N}(t)$ denote the set of all particles alive at time t and let $N(t) := \#\mathscr{N}(t)$. For any

$v \in \mathscr{N}(t)$ let $X_v(t)$ be the position of v at time t ; and for any $s < t$, let $X_v(s)$ be the position of the unique ancestor of v that was alive at time s. We define

$$R(t) := \max_{u \in \mathscr{N}(t)} X_u(t) \text{ and } L(t) := \min_{u \in \mathscr{N}(t)} X_u(t),$$

which stand for the rightmost and leftmost positions, respectively.

The positions of the extremal particles of a BBM, $R(t)$, have been much studied both analytically and probabilistically. Fisher [68] and Kolmogorov et al. [98] introduced the F-KPP equation to which $u(x,t) := \mathbb{P}(R(t) \geq x)$ is a solution. The work of [98] on the traveling wave solutions to the F-KPP equation actually implies that $R(t)/t$ converges almost surely to $\sqrt{2}$. Bramson [44] [45] showed that $R(t) - \sqrt{2}t + (3/2\sqrt{2})\log t$ converges in law. These results hold as well for a wide class of branching random walks under mild conditions : see for example Biggins [34], Addario-Berry and Reed [2], Hu and Shi [87], Aïdékon [4]. In particular, we state the following fact, which is first given by [87] for branching random walks, and is recently proved by Roberts [128] :

$$\liminf_{t\to\infty} \frac{R(t) - \sqrt{2}t}{\log t} = -\frac{3}{2\sqrt{2}} \quad \text{almost surely;} \tag{3.1.1}$$

$$\limsup_{t\to\infty} \frac{R(t) - \sqrt{2}t}{\log t} = -\frac{1}{2\sqrt{2}} \quad \text{almost surely.} \tag{3.1.2}$$

In [101], Lalley and Sellke showed the following interesting property : every particle born in a BBM has a descendant reaching the rightmost position at some future time. Such a particle was thought of having a prominent descendant "in the lead" at this time. This property is in agreement with the branching-selection particle systems investigated in the articles [47], [48] and [25]. These papers bring out the fact that the extremal positions of a branching system on the line cannot always be occupied by the descendants of some "elite" particles.

In the present work, we give some quantitative understanding of this behavior, and precisely speaking, about how long we have to wait so that every particle alive at time s has a descendent that has occupied the rightmost position.

3.1.2 The main problem

Let us make an analytic presentation for our problem. For any $s > 0$ and each particle $u \in \mathscr{N}(s)$, the shifted subtree generated by u is

$$\mathscr{N}^u(t) := \left\{ v \in \mathscr{N}(t+s), u \leq v \right\}, \quad \forall t \geq 0, \tag{3.1.3}$$

where $u \leq v$ indicates that v is a descendant of u or is u itself. Further, for any $v \in \mathscr{N}^u(t)$, let

$$X_v^u(t) := X_v(t+s) - X_u(s), \tag{3.1.4}$$

be its shifted position. We set $R^u(t) := \max_{v \in \mathscr{N}^u(t)} X_v^u(t)$ and $L^u(t) := \min_{v \in \mathscr{N}^u(t)} X_v^u(t)$. Moreover, Let $\left\{\mathscr{F}_t; t \geq 0\right\}$ be the natural filtration of the branching Brownian motion. The branching property implies that, given $\mathscr{F}_s$, $\{R^u(\cdot); u \in \mathscr{N}(s)\}$ are independent copies of $R(\cdot)$. Moreover, we denote by $\mathscr{F}_\infty^u$ the sigma-field generated by the shifted subtree started from the time s rooted at u.

For every $u \in \mathscr{N}(s)$, let

$$\tau_u := \inf\{t > 0 : R(t+s) = X_u(s) + R^u(t)\}. \tag{3.1.5}$$

The random variable τ_u stands for the first time that started from time s, the particle u has a descendant reaching the rightmost position in the system. It is the object in which we are interested. We define

$$\Theta_s := \max_{u \in \mathscr{N}(s)} \tau_u, \tag{3.1.6}$$

which represents the first time when every particle in $\mathscr{N}(s)$ has had a descendant occupying the rightmost position.

According to Lalley and Sellke [101], for any $s > 0$, $\mathbb{P}[\Theta_s < \infty] = 1$. Since $\Theta_s \to \infty$ almost surely as $s \to \infty$, we intend to determine the rate at which Θ_s increases to infinity.

3.1.3 The main results

To estimate $\Theta_s = \max_{u \in \mathscr{N}(s)} \tau_u$, an intuitive idea consists in saying that, the further a particle is away from the rightmost one, the longer it has to wait for a descendant to be located on the rightmost position. We thus first focus on the leftmost particle. Let $\ell(s)$ be the leftmost particle alive at time s. By (3.1.5), $\tau_{\ell(s)}$ is defined as the shortest time needed for $\ell(s)$ to wait to have a descendant occupying the rightmost position.

Theorem 3.1.1. *The following convergence holds almost surely,*

$$\lim_{s \to \infty} \frac{\log \tau_{\ell(s)}}{s} = 4. \tag{3.1.7}$$

However, the leftmost particle is not the one who "drags the feet" of the whole population $\mathscr{N}(s)$. By considering the positions of all particles alive at time s, as well as their evolutions, we obtain our main result as follows.

Theorem 3.1.2. *The following convergence holds almost surely,*

$$\lim_{s\to\infty} \frac{\log \Theta_s}{s} = 2 + 2\sqrt{2} > 4. \tag{3.1.8}$$

Remark 3.1.3. *The proof of the theorems will reveal that the largest τ_u for $u \in \mathscr{N}(s)$ is achieved by some particle located at a position around $-(2-\sqrt{2})s$ which does not split until time $s + \frac{1}{\sqrt{2}}s$ and moves towards to the left as far as possible.*

The rest of this paper is organized as follows. Section 3.2 is devoted to discussing the behaviors of the extremal position $R(\cdot)$, which leads to two propositions. In Section 3.3, we consider the case of two independent branching Brownian motions and state another proposition. We prove Theorem 3.1.1 in Section 3.4 by means of these propositions. Finally, in Section 3.5, we prove Theorem 3.1.2.

3.2 The behavior of the rightmost position

In this section, we study the behaviors of $R(\cdot)$ by comparing $R(t)$ with $m(t) := \sqrt{2}t - \frac{3}{2\sqrt{2}}\log t$.

We recall Proposition 3 in Bramson's work [44]. It is shown that for all $0 \le y \le t^{1/2}$ and $t \ge 2$, there exists a positive constant c which is independent of t and y, such that

$$\mathbb{P}\Big[R(t) > m(t) + y\Big] \le c(1+y)^2 \exp(-\sqrt{2}y). \tag{3.2.1}$$

Therefore, with $c_1 := c + 1$, we get the following inequality, which will be applied several times in our arguments.

Fact 3.2.1 (Bramson [44])**.** *For any $t \ge 2$ and $y \le \sqrt{t}$,*

$$\mathbb{P}\Big[R(t) > m(t) + y\Big] \le c_1\Big(1 + y_+\Big)^2 e^{-\sqrt{2}y}, \tag{3.2.2}$$

with $y_+ := \max\{y, 0\}$.

Let $(B_s; s \ge 0)$ be a standard Brownian motion on $\mathbb{R}$. For any $t > 0$, let $C([0,t],\mathbb{R})$ be the space of continuous functions on $[0,t]$, equipped with the uniform topology (see Chapter 2 of Billingsley [42]). We state the following lemma, which can be found in several papers (e.g. [110] [78]). It is also of frequent use.

Lemma 3.2.2 (many-to-one). *For each $t > 0$ and any measurable function $F : C([0,t], \mathbb{R}) \to \mathbb{R}_+$,*

$$\mathbb{E}\Big[\sum_{u\in\mathcal{N}(t)} F(X_u(s), s\in[0,t])\Big] = e^t \mathbb{E}\Big[F(B_s, s\in[0,t])\Big], \tag{3.2.3}$$

where, for each $u \in \mathcal{N}(t)$ and $s \in [0,t]$, $X_u(s)$ denotes the position, at time s, of the ancestor of u.

Let us present the following inequality as well, which is Equation (57) in [44].

Fact 3.2.3. *For any $s \geq 1$ and any $a > 0$,*

$$\Big(1-\frac{s}{a^2}\Big)\sqrt{\frac{s}{2\pi}}a^{-1}\exp\Big(-\frac{a^2}{2s}\Big) \leq \mathbb{P}[B_s \geq a] \leq \sqrt{\frac{s}{2\pi}}a^{-1}\exp\Big(-\frac{a^2}{2s}\Big).$$

It immediately follows that

$$\mathbb{P}[B_s \leq -a] = \mathbb{P}[B_s \geq a] \leq \frac{\sqrt{s}}{a}\exp\Big(-\frac{a^2}{2s}\Big). \tag{3.2.4}$$

Moreover, if $a = \alpha s$ with some constant $\alpha > 0$, we have

$$\mathbb{P}[B_s \leq -\alpha s] = \mathbb{P}[B_s \geq \alpha s] = \exp\Big\{-\Big(\frac{\alpha^2}{2}+o_s(1)\Big)s\Big\}, \tag{3.2.5}$$

where $o_s(1) \to 0$ as s goes to infinity.

We define, for any $y > 0$,

$$T(y) := \inf\Big\{t \geq 1; R(t) - m(t) > y\Big\}.$$

Because of (3.1.2), one immediately sees that $\mathbb{P}\Big[T(y) < \infty\Big] = 1$ for any $y > 0$. Moreover, $T(y) \uparrow \infty$ almost surely as $y \uparrow \infty$.

Proposition 3.2.4. *The following convergence holds almost surely,*

$$\lim_{y\to\infty} \frac{\log T(y)}{y} = \sqrt{2}. \tag{3.2.6}$$

Proof. First, we prove the lower bound.

Let $2 \leq y \leq \sqrt{t}$, and set

$$\Lambda := \mathbb{E}\Big[\int_1^{t+1} \mathbf{1}_{(R(s)>m(s)+y-1)} ds\Big].$$

Clearly, $\Lambda = \int_1^{t+1} \mathbb{P}\Big[R(s) > m(s) + y - 1\Big] ds$. Hence,

$$\begin{aligned}\Lambda &= \int_1^{y^2} \mathbb{P}\Big[R(s) > m(s) + y - 1\Big] ds + \int_{y^2}^{t+1} \mathbb{P}\Big[R(s) > m(s) + y - 1\Big] ds \\ &\leq \int_1^{y^2} \mathbb{E}\Big[\sum_{u \in \mathcal{N}(s)} 1_{(X_u(s) > m(s)+y-1)}\Big] ds + \int_{y^2}^{t+1} \mathbb{P}\Big[R(s) > m(s) + y - 1\Big] ds.\end{aligned}$$

By the many-to-one lemma and by (3.2.4),

$$\begin{aligned}&\int_1^{y^2} \mathbb{E}\Big[\sum_{u \in \mathcal{N}(s)} 1_{(X_u(s) > m(s)+y-1)}\Big] ds = \int_1^{y^2} e^s \mathbb{P}\Big[B_s > m(s) + y - 1\Big] ds \\ &\leq \int_1^{y^2} \frac{\sqrt{s}}{m(s) + y - 1} \exp\Big\{\frac{-(m(s)+y-1)^2}{2s} + s\Big\} ds. \qquad (3.2.7)\end{aligned}$$

Note that for $m(s) = \sqrt{2}s - \frac{3}{2\sqrt{2}} \log s$ with $s \in [1, y^2]$, the inequalities

$$m(s) + y - 1 \geq \sqrt{2}s \text{ and } \exp\Big(\frac{-(m(s)+y-1)^2}{2s} + s\Big) \leq s^{3/2} e^{-\sqrt{2}(y-1)} \qquad (3.2.8)$$

hold. Plugging them into the integration of (3.2.7) yields that

$$\int_1^{y^2} \mathbb{E}\Big[\sum_{u \in \mathcal{N}(s)} 1_{(X_u(s) > m(s)+y-1)}\Big] ds \leq \int_1^{y^2} \frac{s^2}{\sqrt{2}s} e^{-\sqrt{2}(y-1)} ds \leq c_2 y^4 e^{-\sqrt{2}y}, \qquad (3.2.9)$$

which is then bounded by $c_2 t y^2 e^{-\sqrt{2}y}$ as $y \leq \sqrt{t}$. Meanwhile, by the inequality (3.2.2),

$$\int_{y^2}^{t+1} \mathbb{P}\Big[R(s) > m(s) + y - 1\Big] ds \leq \int_{y^2}^{t+1} c_1 y^2 e^{-\sqrt{2}y + \sqrt{2}} ds \leq c_1 t y^2 e^{-\sqrt{2}y + \sqrt{2}}. \qquad (3.2.10)$$

Combining (3.2.9) with (3.2.10), we have

$$\Lambda \leq c_3 t y^2 e^{-\sqrt{2}y}. \qquad (3.2.11)$$

with $c_3 > 0$ a constant independent of (y, t).

On the other hand,

$$\begin{aligned}\Lambda &\geq \mathbb{E}\Big[\int_1^{t+1} \mathbf{1}_{(R(s) > m(s)+y-1)} ds;\, T(y) \leq t\Big] \\ &= \int_1^t \mathbb{P}\Big[T(y) \in dr\Big] \mathbb{E}\Big[\int_1^{t+1} \mathbf{1}_{(R(s) > m(s)+y-1)} ds \Big| T(y) = r\Big].\end{aligned}$$

Conditionally on the event $\{T(y) = r \leq t\}$, the rightmost particle in $\mathscr{N}(r)$, denoted by ω, is located at $m(r)+y$. Started from the time r, ω moves according to a Brownian motion and splits into two after an exponential time. By ignoring its branches, we observe that $\Big\{R(s+r) > [m(r)+y] + [\sqrt{2}s-1] \geq m(s+r)+y-1\Big\}$ is satisfied as long as the Brownian motion realized by ω keeps lying above $\sqrt{2}s-1$. Hence, given $\{T(y) = r \leq t\}$,

$$\int_1^{t+1} \mathbf{1}_{\left(R(s)>m(s)+y-1\right)} ds \geq_{st} \int_0^{t+1-r} \mathbf{1}_{\left(B_s>\sqrt{2}s-1\right)} ds \geq \min\{1, T_{-1}^{(-\sqrt{2})}\},$$

where $\geq_{st}$ denotes stochastic dominance and $T_{-1}^{(-\sqrt{2})} := \inf\{t \geq 0; B_t < \sqrt{2}t-1\}$.

These arguments imply that

$$\begin{aligned} \Lambda \ &\geq \textstyle\int_1^t \mathbb{P}[T(y) \in dr]\mathbb{E}\Big[\min\{1, T_{-1}^{(-\sqrt{2})}\}\Big] \\ &=: c_{13}\mathbb{P}[T(y) \leq t], \end{aligned} \tag{3.2.12}$$

where $c_{13} := \mathbb{E}\Big[\min\{1, T_{-1}^{(-\sqrt{2})}\}\Big] \in (0,\infty)$. Compared with (3.2.11), this tells us that

$$\mathbb{P}\Big[T(y) \leq t\Big] \leq c_5 t y^2 e^{-\sqrt{2}y}, \text{ for } 2 \leq y \leq \sqrt{t}, \tag{3.2.13}$$

where $c_5 := \frac{c_3}{c_{13}} \in (0,\infty)$.

Taking $t = e^{\sqrt{2}y(1-\delta)}$ with $\delta \in (0,1)$ yields that

$$\sum_{k=1}^{\infty} \mathbb{P}\Big[T(k) \leq e^{\sqrt{2}k(1-\delta)}\Big] < \infty.$$

According to the Borel-Cantelli lemma,

$$\liminf_{y\to\infty} \frac{\log T(y)}{y} \geq \sqrt{2}, \text{ almost surely,}$$

proving the lower bound in the proposition.

To prove the upper bound, we recall that

$$R^u(t) = \max\{X_v^u(t); v \in \mathscr{N}^u(t)\},\ u \in \mathscr{N}(s).$$

Obviously, $R^u(t); u \in \mathscr{N}(s)$ are i.i.d. given $\mathscr{F}_s$, and are distributed as $R(t)$.

We fix $a_y \in (0,y)$ and define the measurable events

$$\begin{aligned} \Sigma_1 \ &:= \ \left\{L(a_y) \geq -2a_y; N(a_y) \geq \exp\left(\frac{1}{2}a_y\right)\right\}, \\ \Sigma \ &:= \ \Sigma_1 \cap \Big\{T(y) > e^{\sqrt{2}y(1+\delta)}\Big\}. \end{aligned}$$

Then,

$$\begin{aligned}\mathbb{P}\Big[T(y) > e^{\sqrt{2}y(1+\delta)}\Big] &\leq \mathbb{P}\Big[\Sigma_1^c\Big] + \mathbb{P}\Big[\Sigma_1 \cap \Big\{T(y) > e^{\sqrt{2}y(1+\delta)}\Big\}\Big] \\ &\leq \mathbb{P}\Big[N(a_y) \leq \exp\Big(\tfrac{1}{2}a_y\Big)\Big] + \mathbb{P}\Big[L(a_y) \leq -2a_y\Big] + \mathbb{P}\Big[\Sigma\Big].\end{aligned} \tag{3.2.14}$$

We choose $a_y = \frac{\delta}{4+2\sqrt{2}}y =: \delta_1 y$ from now on to evaluate $\mathbb{P}[\Sigma]$. Since $\Sigma_1 \in \mathscr{F}_{a_y}$, for y large enough so that $2e^{\sqrt{2}y(1+\frac{1}{2}\delta)} \leq e^{\sqrt{2}y(1+\delta)} - a_y$, we have

$$\begin{aligned}\mathbb{P}\Big[\Sigma\Big|\mathscr{F}_{a_y}\Big] &\leq \mathbf{1}_{\Sigma_1} \prod_{u\in\mathscr{N}(a_y)} \mathbb{P}\Big[R^u(r) \leq m(a_y+r) + y - X_u(a_y), \forall r \leq e^{\sqrt{2}y(1+\delta)} - a_y \Big| \mathscr{F}_{a_y}\Big] \\ &\leq \mathbb{P}\Big[R(r) \leq m(r) + y + 2a_y + \sqrt{2}a_y, \forall r \in \Big[e^{\sqrt{2}y(1+\frac{1}{2}\delta)}, 2e^{\sqrt{2}y(1+\frac{1}{2}\delta)}\Big]\Big]^{e^{a_y/2}} \\ &\leq \mathbb{P}\Big[R(r) \leq m(r) + \frac{1}{\sqrt{2}}\log r, \forall r \in \Big[e^{\sqrt{2}y(1+\frac{1}{2}\delta)}, 2e^{\sqrt{2}y(1+\frac{1}{2}\delta)}\Big]\Big]^{e^{a_y/2}}.\end{aligned}$$

At this stage, it is convenient to recall the proof of Proposition 15 of [128], saying that there exists a constant $c' > 0$ such that for y large enough,

$$\mathbb{P}\Big[\exists r \in \Big[e^{\sqrt{2}y(1+\frac{1}{2}\delta)}, 2e^{\sqrt{2}y(1+\frac{1}{2}\delta)}\Big] : R(r) \geq m(r) + \frac{1}{\sqrt{2}}\log r\Big] > c' > 0.$$

Thus, $\mathbb{P}\Big[\Sigma\Big] \leq (1-c')^{e^{a_y/2}} \leq \exp(-c'e^{\delta_1 y/2})$.

It remains to estimate $\mathbb{P}\big[N(a_y) \leq \exp(\frac{1}{2}a_y)\big]$ and $\mathbb{P}\big[L(a_y) \leq -2a_y\big]$. On the one hand, the branching mechanism tells us that for any $s \geq 0$, $N(s)$ follows the geometric distribution with parameter e^{-s} (for example, see Page 324 of [120]). It thus yields that $\mathbb{P}\big[N(a_y) \leq \exp(\frac{1}{2}a_y)\big] \leq e^{-\delta_1 y/2}$. On the other hand, as shown in Proposition 1 of Lalley and Sellke [103], for any $\mu \geq \sqrt{2}$ and $s > 0$,

$$\mathbb{P}\Big[L(s) \leq -\mu s\Big] = \mathbb{P}\Big[R(s) \geq \mu s\Big] \leq \mu^{-1}(2\pi s)^{-1/2} \exp\Big(-s\Big(\frac{\mu^2}{2} - 1\Big)\Big). \tag{3.2.15}$$

Consequently, (3.2.14) becomes that

$$\begin{aligned}\mathbb{P}\Big[T(y) > e^{\sqrt{2}y(1+\delta)}\Big] &\leq e^{-\delta_1 y/2} + e^{-\delta_1 y} + \exp(-c'e^{\delta_1 y/2}) \\ &\leq c_6 e^{-\delta_1 y/2}.\end{aligned} \tag{3.2.16}$$

By the Borel-Cantelli lemma again, we conclude that almost surely,

$$\limsup_{y\to\infty} \frac{\log T(y)}{y} \leq \sqrt{2},$$

which completes the proof of the proposition. □

For $\alpha > 0$ and $\beta > 0$, set

$$p(z,\alpha,\beta) := \mathbb{P}\Big[\exists r \le e^{\alpha z} : R(r) \le m(r) - \beta z\Big].$$

Proposition 3.2.5. *There exists a positive constant C_1, independent of (α,β,z), such that for any $z \ge z(\alpha,\beta)$,*

$$p(z,\alpha,\beta) \le C_1 \exp\Big(-\frac{\beta z}{6\sqrt{2}}\Big). \tag{3.2.17}$$

Proof. It follows from (3.1.1) that as $z \to \infty$,

$$p(z,\alpha,\beta) \;=\; \mathbb{P}\Big[\exists r \le e^{\alpha z} : R(r) \le m(r) - \beta z\Big] \longrightarrow 0.$$

Hence, there exists $z_0(\alpha,\beta)$ large enough, such that for all $z \ge z_0(\alpha,\beta)$,

$$\mathbb{P}\Big[\exists r \le e^{\alpha z} : R(r) \le m(r) - \beta z/2\Big] \le 1/2. \tag{3.2.18}$$

For any $b_z < e^{\alpha z}$, we have

$$\begin{aligned} p(z,\alpha,\beta) \le \mathbb{P}\Big[\exists u \in \mathcal{N}(b_z), s.t.\ \min_{s \le b_z} X_u(s) \le \sqrt{2} b_z - \beta z/2\Big] \\ + \mathbb{P}\Big[\Big\{\exists r \le e^{\alpha z} : R(r) \le m(r) - \beta z\Big\} \cap \Big\{L(b_z) \ge \sqrt{2} b_z - \beta z/2\Big\}\Big]. \end{aligned} \tag{3.2.19}$$

On the one hand, by the many-to-one lemma,

$$\begin{aligned} \mathbb{P}\Big[\exists u \in \mathcal{N}(b_z), s.t.\ \min_{s \le b_z} X_u(s) \le \sqrt{2} b_z - \beta z/2\Big] &\le \mathbb{E}\Big[\sum_{u \in \mathcal{N}(b_z)} \mathbf{1}_{(\min_{s \le b_z} X_u(s) \le \sqrt{2} b_z - \beta z/2)}\Big] \\ &= e^{b_z} \mathbb{P}\Big[\min_{s \le b_z} B_s \le \sqrt{2} b_z - \beta z/2\Big]. \end{aligned}$$

On the other hand, by simple observations,

$$\begin{aligned} &\mathbb{P}\Big[\Big\{\exists r \le e^{\alpha z} : R(r) \le m(r) - \beta z\Big\} \cap \Big\{L(b_z) \ge \sqrt{2} b_z - \beta z/2\Big\}\Big] \\ \le\ & \mathbb{P}\Big[\bigcap_{u \in \mathcal{N}(b_z)} \Big\{\exists t \le e^{\alpha z}, s.t.\ R^u(t) < m(t) - \beta z/2\Big\}\Big] \\ =\ & \mathbb{E}\Big[\prod_{u \in \mathcal{N}(b_z)} \mathbb{P}\Big[\exists t \le e^{\alpha z}, s.t.\ R(t) < m(t) - \beta z/2\Big]\Big], \end{aligned}$$

where the last equality follows from the branching property. Going back to (3.2.19), one has

$$p(z,\alpha,\beta) \le e^{b_z}\mathbb{P}\Big[\min_{s\le b_z} B_s \le \sqrt{2}b_z - \beta z/2\Big] + \mathbb{E}\Big[\prod_{u\in\mathcal{N}(b_z)} \mathbb{P}\Big[\exists t \le e^{\alpha z}, s.t.\ R(t) < m(t) - \beta z/2\Big]\Big].$$

Let $b_z = \frac{\beta}{6\sqrt{2}}z$. Then, by (3.2.18), for all $z \ge z(\alpha,\beta) := \max\{z_0(\alpha,\beta), \frac{1}{\beta}\}$,

$$\begin{aligned} p(z,\alpha,\beta) &\le e^{b_z}\mathbb{P}\Big[\min_{s\le b_z} B_s \le -\beta z/3\Big] + \mathbb{E}\Big[\Big(\frac{1}{2}\Big)^{N(b_z)}\Big] \\ &\le c_7 e^{-3b_z} + e^{-b_z} \le C_1 \exp\Big(-\frac{\beta z}{6\sqrt{2}}\Big), \end{aligned}$$

with $C_1 := c_7 + 1$, which completes the proof of the proposition. □

Corollary 3.2.6. *For any $\delta \in (0,1)$, there exists some $s(\delta) \ge 1$, such that for all $s \ge s(\delta)$,*

$$\mathbb{P}\Big[R(s) \le \sqrt{2}(1-\delta)s\Big] \le C_1 \exp\Big(-\frac{\delta s}{12\sqrt{2}}\Big). \tag{3.2.20}$$

Proof. Since we always have $m(s) - \delta s/2 \ge \sqrt{2}(1-\delta)s$ when s is sufficiently large,

$$\mathbb{P}\Big[R(s) \le \sqrt{2}(1-\delta)s\Big] \le \mathbb{P}\Big[\exists r \le e^s : R(r) \le m(r) - \delta s/2\Big].$$

which by Proposition 3.2.5 is bounded by $C_1 \exp\big(-\frac{\delta s}{12\sqrt{2}}\big)$ for all s large enough. □

3.3 The case of two independent branching Brownian motions

We consider two independent branching Brownian motions, denoted by $\mathbb{X}^A(\cdot)$ and $\mathbb{X}^B(\cdot)$. Suppose that $\mathbb{P}[X^A(0) = 0] = \mathbb{P}[X^B(0) = z] = 1$ with $z > 0$, where $\mathbb{X}^A(0)$ and $\mathbb{X}^B(0)$ represent the position of the roots, respectively. We write $R^A(\cdot)$ ($R^B(\cdot)$, respectively) for the position of rightmost particle of the BBM $\mathbb{X}^A(\cdot)$ ($\mathbb{X}^B(\cdot)$, respectively). We define, for any $y > 0$,

$$\begin{aligned} T^A(y) &:= \inf\{t \ge 1; R^A(t) > m(t) + y\}; \\ T^B(y) &:= \inf\{t \ge 1; R^B(t) > m(t) + y\}. \end{aligned}$$

Let $T^{A>B}$ be the first time when the rightmost point of $\mathbb{X}^A$ exceeds that of $\mathbb{X}^B$, i.e.,

$$T^{A>B} = T^{A>B}(z) := \inf\{t \ge 0; R^A(t) > R^B(t)\}.$$

We immediately observe that the distribution of $T^{A>B}(z)$ merely depends on the parameter z. Actually, we can take another pair of independent standard BBM's (both rooted at the origin), namely, $\mathbb{X}^I(\cdot)$ and $\mathbb{X}^{II}(\cdot)$. Their rightmost positions are denoted by $R^I(\cdot)$ and $R^{II}(\cdot)$, respectively. For any positive z, let

$$\mathscr{T}(z) := \inf\{t \geq 0 : R^I(t) - R^{II}(t) > z\}.$$

Then $T^{A>B}(z)$ is distributed as $\mathscr{T}(z)$. Besides, $z \mapsto \mathscr{T}(z)$ is increasing.

Proposition 3.3.1. *The following convergence holds almost surely,*

$$\lim_{z\to\infty} \frac{\log \mathscr{T}(z)}{z} = \sqrt{2}. \tag{3.3.1}$$

Proof. For any $\delta \in (0,1)$,

$$\mathbb{P}\left[\mathscr{T}(z) \leq e^{\sqrt{2}z(1-\delta)}\right] = \mathbb{P}\left[T^{A>B}(z) \leq e^{\sqrt{2}z(1-\delta)}\right] \leq p_1 + p_2,$$

where

$$\begin{aligned} p_1 &:= \mathbb{P}\left[\exists t \leq e^{\sqrt{2}z(1-\delta)}, s.t.\ R^B(t) < m(t) + z - \delta z/2\right], \\ p_2 &:= \mathbb{P}\left[\left\{T^{A>B} \leq e^{\sqrt{2}z(1-\delta)}\right\} \cap \left\{R^B(t) \geq m(t) + (1-\delta/2)z, \forall t \leq e^{\sqrt{2}z(1-\delta)}\right\}\right]. \end{aligned}$$

Clearly, $p_1 = \mathbb{P}\left[\exists t \leq e^{\sqrt{2}z(1-\delta)}, s.t.\ R(t) < m(t) - \delta z/2\right] = p(z, \sqrt{2}(1-\delta), \delta/2)$. By Proposition 3.2.5, for all $z \geq z(\delta)$,

$$p_1 \leq C_1 \exp\left(-\frac{\delta z}{12\sqrt{2}}\right).$$

At the same time, we notice that

$$\begin{aligned} &\left\{T^{A>B} \leq e^{\sqrt{2}z(1-\delta)}\right\} \cap \left\{R^B(t) \geq m(t) + (1-\delta/2)z, \forall t \leq e^{\sqrt{2}z(1-\delta)}\right\} \subset \\ &\quad \left\{\exists t \leq e^{\sqrt{2}z(1-\delta)} : R^A(t) \geq R^B(t) \geq m(t) + (1-\delta/2)z\right\} \subset \left\{T^A\Big((1-\delta/2)z\Big) \leq e^{\sqrt{2}z(1-\delta)}\right\}. \end{aligned} \tag{3.3.2}$$

This yields that

$$p_2 \leq \mathbb{P}\left[T\Big((1-\delta/2)z\Big) \leq e^{\sqrt{2}z(1-\delta)}\right] \leq c_5 z^2 e^{-\delta z/\sqrt{2}},$$

because of the inequality (3.2.13).

As a result,

$$\mathbb{P}\left[\mathscr{T}(z) \leq e^{\sqrt{2}z(1-\delta)}\right] \leq C_1 \exp\left(-\frac{\delta z}{12\sqrt{2}}\right) + c_5 z^2 e^{-\delta z/\sqrt{2}} \leq c_8 \exp\left(-\frac{\delta z}{12\sqrt{2}}\right), \tag{3.3.3}$$

for some constant $c_8 > 0$ and all z large enough. Thus, by the Borel-Cantelli lemma,

$$\liminf_{z\to\infty} \frac{\log \mathscr{T}(z)}{z} \geq \sqrt{2}, \text{ almost surely.}$$

To prove the upper bound, we observe that

$$\mathbb{P}\left[\mathscr{T}(z) > e^{\sqrt{2}z(1+\delta)}\right] = \mathbb{P}\left[T^{A>B}(z) > e^{\sqrt{2}z(1+\delta)}\right] \leq q_1 + q_2, \tag{3.3.4}$$

where

$$\begin{aligned} q_1 &:= \mathbb{P}\left[\left\{T^A\Big(z(1+\delta/2)\Big) > e^{\sqrt{2}z(1+\delta)}\right\} \cup \left\{T^A\Big(z(1+\delta/2)\Big) < e^{\sqrt{2}z}\right\}\right], \\ q_2 &:= \mathbb{P}\left[e^{\sqrt{2}z} \leq T^A\Big(z(1+\delta/2)\Big) \leq e^{\sqrt{2}z(1+\delta)} < T^{A>B}(z)\right]. \end{aligned}$$

Notice that $T^A(y)$ is distributed as $T(y)$ for any $y > 0$. According to the inequalities (3.2.13) and (3.2.16), there exists $\delta_2 := \delta_2(\delta) > 0$ such that $q_1 \leq e^{-\delta_2 z}$ for z large enough. It remains to estimate q_2 :

$$\begin{aligned} q_2 &\leq \int_{e^{\sqrt{2}z}}^{e^{\sqrt{2}z(1+\delta)}} \mathbb{P}\left[T^A\Big(z(1+\delta/2)\Big) \in dr\right] \mathbb{P}\left[T^{A>B} > r \,\middle|\, T^A\Big(z(1+\delta/2)\Big) = r\right] \\ &\leq \int_{e^{\sqrt{2}z}}^{e^{\sqrt{2}z(1+\delta)}} \mathbb{P}\left[T^A\Big(z(1+\delta/2)\Big) \in dr\right] \mathbb{P}\left[R^B(r) > m(r) + z(1+\delta/2)\right]. \end{aligned}$$

By the inequality (3.2.2) again, this tells that

$$\begin{aligned} q_2 &\leq \int_{e^{\sqrt{2}z}}^{e^{\sqrt{2}z(1+\delta)}} \mathbb{P}\left[T^A\Big(z(1+\delta/2)\Big) \in dr\right] c_2(z+1)^2 e^{-\sqrt{2}\delta z/2} \\ &\leq c_2(z+1)^2 e^{-\sqrt{2}\delta z/2}. \end{aligned}$$

Thus, recalling (3.3.4), we obtain that for all z large enough,

$$\mathbb{P}\left[\mathscr{T}(z) > e^{\sqrt{2}z(1+\delta)}\right] \leq e^{-\delta_2 z} + c_2(z+1)^2 e^{-\sqrt{2}\delta z/2}.$$

It follows that almost surely $\limsup_{z\to\infty} \frac{\log \mathscr{T}(z)}{z} \leq \sqrt{2}$. Proposition 3.3.1 is proved. □

3.4 Proof of Theorem 3.1.1

Proof. For any $k \in \mathbb{N}_+$ and $\delta \in (0, 1/20)$, we define

$$\mathscr{N}_\delta(k) := \{u \in \mathscr{N}(k) : X_u(k) \le -\sqrt{2}(1-\delta/2)k\}.$$

In order to study the asymptotic behavior of $\tau_{\ell(s)}$ for $s \in \mathbb{R}_+$, we first look for a lower bound for $\min_{u\in\mathscr{N}_\delta(k)} \tau_u$ and an upper bound for $\max_{u\in\mathscr{N}_\delta(k)} \tau_u$.

Recall the definitions (3.1.3) and (3.1.4) of the shifted subtrees. For any particle $u \in \mathscr{N}_\delta(k)$, we use $\mathbb{X}^u(\cdot)$ to represent the branching Brownian motion generated by u started from the time k. Meanwhile, we use $\mathbb{X}^r(\cdot)$ to represent the branching Brownian motion generated by the rightmost point at time k. Accordingly, the random variable $T^{u>r}$ is defined to be the first time when u has a descendant exceeding all descendants of the rightmost particle at time k.

Considering that $T^{u>r} \le \tau_u$ for each $u \in \mathscr{N}_\delta(k)$, one sees that

$$\mathbb{P}\Big[\bigcup_{u\in\mathscr{N}_\delta(k)} \{\tau_u \le e^{4k(1-10\delta)}\}\Big] \le p_1' + p_2',$$

where

$$\begin{aligned} p_1' &:= \mathbb{P}\Big[R(k) \le \sqrt{2}(1-\delta/2)k\Big], \\ p_2' &:= \mathbb{E}\Big[1_{\left(R(k)\ge\sqrt{2}(1-\delta/2)k\right)} \sum_{u\in\mathscr{N}_\delta(k)} 1_{\left(T^{u>r}\le e^{4k(1-10\delta)}\right)}\Big]. \end{aligned}$$

Given $\mathscr{F}_k$, the BBM's $\mathbb{X}^u$ and $\mathbb{X}^r$ are independent. Then,

$$\begin{aligned} p_2' &\le \mathbb{E}\Big[\sum_{u\in\mathscr{N}_\delta(k)} 1_{\left(R(k)\ge\sqrt{2}(1-\delta/2)k\right)} \mathbb{P}\Big[T^{u>r} \le e^{4k(1-10\delta)}\Big|\mathscr{F}_k\Big]\Big] \\ &= \mathbb{E}\Big[\sum_{u\in\mathscr{N}_\delta(k)} 1_{\left(R(k)\ge\sqrt{2}(1-\delta/2)k\right)} \mathbb{P}\Big[\mathscr{T}(R(k)-X_u(k)) \le e^{4k(1-10\delta)}\Big|\mathscr{F}_k\Big]\Big]. \end{aligned}$$

By the monotonicity of $\mathscr{T}(\cdot)$, this gives that

$$\begin{aligned} p_2' &\le \mathbb{E}\Big[\sum_{u\in\mathscr{N}_\delta(k)} \mathbb{P}\Big[\mathscr{T}(2\sqrt{2}k(1-\delta/2)) \le e^{4k(1-10\delta)}\Big]\Big] \\ &= \mathbb{E}\Big[\sum_{u\in\mathscr{N}_\delta(k)} 1\Big]\mathbb{P}\Big[\mathscr{T}(2\sqrt{2}k(1-\delta/2)) \le e^{4k(1-10\delta)}\Big]. \end{aligned}$$

Using the inequality (3.3.3), for all k sufficiently large,

$$p_2' \le \mathbb{E}\Big[\sum_{u\in\mathscr{N}_\delta(k)} 1\Big] c_8 \exp\Big(-\frac{3\delta k}{2}(1-\delta/2)\Big).$$

Then by the many-to-one lemma and by (3.2.4), we obtain that

$$\begin{aligned} p_2' &\leq e^k \mathbb{P}[B_k \leq -\sqrt{2}k(1-\delta/2)] c_8 \exp\Big(-\tfrac{3\delta k}{2}(1-\delta/2)\Big) \\ &\leq e^{-c_9 \delta k}, \end{aligned} \tag{3.4.1}$$

where c_9 is a positive constant independent of (δ, k).

In view of Corollary 3.2.6, for large k, one has

$$p_1' \leq C_1 \exp\left(-\frac{\delta k}{24\sqrt{2}}\right). \tag{3.4.2}$$

Combining (3.4.2) with (3.4.1) yields that for k large enough,

$$\mathbb{P}\Bigg[\bigcup_{u \in \mathscr{N}_\delta(k)} \{\tau_u \leq e^{4k(1-10\delta)}\}\Bigg] \leq C_1 \exp\left(-\frac{\delta k}{24\sqrt{2}}\right) + e^{-c_9\delta k}.$$

By the Borel-Cantelli lemma, almost surely,

$$\liminf_{k\to\infty} \frac{\log \min_{u\in\mathscr{N}_\delta(k)} \tau_u}{k} \geq 4(1-10\delta), \tag{3.4.3}$$

which gives the lower bound for $\min_{u\in\mathscr{N}_\delta(k)} \tau_u$.

To obtain an upper bound for $\max_{u\in\mathscr{N}_\delta(k)} \tau_u$, let us estimate $\mathbb{P}\Big[\cup_{u\in\mathscr{N}_\delta(k)} \Big\{\tau_u \geq e^{4k(1+10\delta)}\Big\}\Big]$. We consider the subtree generated by any particle $u \in \mathscr{N}_\delta(k)$. Recall that the shifted positions of its descendants are denoted by

$$X_v^u(\cdot) := X_v(\cdot + k) - X_u(k) \text{ for any } v \in \mathscr{N}(\cdot + k) \text{ satisfying } u < v,$$

and that $R^u(\cdot) := \max X_v^u(\cdot)$. We set $T^u(y) := \inf\{t \geq 1; R^u(t) - m(t) > y\}$ for any $y > 0$, which is obviously distributed as $T(y)$. Let $y = 2\sqrt{2}k(1+\delta/2)$, then

$$\mathbb{P}\Bigg[\bigcup_{u\in\mathscr{N}_\delta(k)} \Big\{\tau_u \geq e^{4k(1+10\delta)}\Big\}\Bigg] \leq q_1' + q_2' + q_3', \tag{3.4.4}$$

where

$$\begin{aligned} q_1' &:= \mathbb{P}\Bigg[\bigcup_{u\in\mathscr{N}_\delta(k)} \Big(\Big\{T^u(y) \geq e^{4k(1+10\delta)}\Big\} \cup \Big\{T^u(y) \leq e^k\Big\}\Big)\Bigg], \\ q_2' &:= \mathbb{P}\Big[L(k) \leq -\sqrt{2}k\Big], \\ q_3' &:= \mathbb{P}\Bigg[\bigcup_{u\in\mathscr{N}_\delta(k)} \{e^k < T^u(y) < e^{4k(1+10\delta)} \leq \tau_u\}; L(k) > -\sqrt{2}k\Bigg]. \end{aligned}$$

First, we observe that

$$q_1' \leq \mathbb{E}\left[\sum_{u\in\mathcal{N}_\delta(k)} 1\right]\mathbb{P}\left[\left\{T(y)\geq e^{4k(1+10\delta)}\right\}\cup\left\{T(y)\leq e^k\right\}\right].$$

Using the many-to-one lemma for the first term on the right-hand side,

$$q_1' \leq e^k\mathbb{P}\left[B_k\leq -\sqrt{2}k(1-\delta/2)\right]\mathbb{P}\left[\left\{T(y)\geq e^{4k(1+10\delta)}\right\}\cup\left\{T(y)\leq e^k\right\}\right].$$

According to the inequalities (3.2.4) (3.2.13) and (3.2.16), there exists $\delta_4 := \delta_4(\delta) > 0$ such that $q_1' \leq e^{-\delta_4 k}$ for k large enough. Meanwhile, by (3.2.2), $q_2' \leq 2c_2(\log k+1)^2 k^{-3/2}$.

It remains to bound q_3'. Since $T^u(y)$ is independent of $\mathscr{F}_k$, it follows that

$$q_3' \;\leq\; \mathbb{E}\left[\sum_{u\in\mathcal{N}_\delta(k)}\int_{e^k}^{e^{4k(1+10\delta)}}\mathbb{P}\left[T^u(y)\in dr\right]\mathbb{P}\left[\tau_u>r; L(k)\geq -\sqrt{2}k\middle| T^u(y)=r,\mathscr{F}_k\right]\right].$$

Given $\{T^u(y)=r\}$ and $\mathscr{F}_k$, the event $\{\tau_u>r\}\cap\{L(k)\geq-\sqrt{2}k\}$ implies that $\cup_{w\in\mathcal{N}(k)\setminus\{u\}}\{R^\omega(r)+X_w(k)>R^u(r)+X_u(k)\geq m(r)+y-\sqrt{2}k\}$, whose probability is less than $\sum_{w\in\mathcal{N}(k)\setminus\{u\}}c_1\Big(1+(y-\sqrt{2}k-X_w(k))_+^2\Big)e^{-\sqrt{2}y+2k+\sqrt{2}X_w(k)}$ (see (3.2.2)). This yields that

$$\begin{aligned}
q_3' \;&\leq\; \mathbb{E}\left[\sum_{u\in\mathcal{N}_\delta(k)}\int_{e^k}^{e^{4k(1+10\delta)}}\mathbb{P}\left[T^u(y)\in dr\right]\sum_{w\in\mathcal{N}(k)\setminus\{u\}}c_2(y+1)^2e^{-\sqrt{2}y+2k+\sqrt{2}X_w(k)}\right]\\
&\leq\; \mathbb{E}\left[\sum_{u\in\mathcal{N}_\delta(k)}\sum_{w\in\mathcal{N}(k)\setminus\{u\}}c_2(y+1)^2e^{-\sqrt{2}y+2k+\sqrt{2}X_\omega(k)}\right]\\
&=\; c_2(y+1)^2e^{-\sqrt{2}y+2k}\mathbb{E}\left[\sum_{u\in\mathcal{N}_\delta(k)}\sum_{w\in\mathcal{N}(k)\setminus\{u\}}e^{\sqrt{2}X_w(k)}\right].
\end{aligned}$$

By integrating with respect to the last time at which the most recent common ancestor of u and ω was alive (see e.g. [78] for more details), $\mathbb{E}\left[\sum_{u\in\mathcal{N}_\delta(k)}\sum_{\omega\neq u}e^{\sqrt{2}X_\omega(k)}\right]$ is equal to

$$\begin{aligned}
&2\int_0^k e^{2k-s}ds\int_{\mathbb{R}}\mathbb{P}\left[B_s\in dx\right]\mathbb{P}\left[B_k\leq-\sqrt{2}(1-\delta/2)k\middle|B_s=x\right]\mathbb{E}\left[e^{\sqrt{2}B_k}\middle|B_s=x\right]\\
=\;&2\int_0^k e^{2k-s}ds\int_{\mathbb{R}}\mathbb{P}\left[B_s\in dx\right]\mathbb{P}\left[B_k\leq-\sqrt{2}(1-\delta/2)k\middle|B_s=x\right]e^{\sqrt{2}x}e^{k-s},
\end{aligned}$$

where the second equivalence follows from the Markov property of Brownian Motion. We rearrange the integration as follows :

$$\begin{aligned}\mathbb{E}\Bigg[\sum_{u\in\mathcal{N}_\delta(k)}\sum_{w\in\mathcal{N}(k)\setminus\{u\}}e^{\sqrt{2}X_w(k)}\Bigg] &= 2\int_0^k e^{3k-2s}\mathbb{E}\Big[e^{\sqrt{2}B_s};B_k\le-\sqrt{2}(1-\delta/2)k\Big]ds\\ &= 2\int_0^k e^{3k-2s}\mathbb{E}\Big[e^{\sqrt{2}B_k}e^{-\sqrt{2}(B_k-B_s)};B_k\le-\sqrt{2}(1-\delta/2)k\Big]ds\\ &\le 2\int_0^k e^{3k-2s}e^{-2(1-\delta/2)k}\mathbb{E}\Big[e^{-\sqrt{2}(B_k-B_s)}\Big]ds,\end{aligned}$$

which is bounded by $e^{(2+\delta)k}$ by simple computation. Thus, $q_3'\le c_{10}k^2e^{-\delta k}$ for some constant $c_{10}>0$.

Going back to (3.4.4),

$$\mathbb{P}\Bigg[\bigcup_{u\in\mathcal{N}_\delta(k)}\Big\{\tau_u\ge e^{4k(1+10\delta)}\Big\}\Bigg]\le e^{-\delta_4 k}+2c_2(\log k+1)^2k^{-3/2}+c_{10}k^2e^{-\delta k},$$

for all k sufficiently large.

Therefore, by the Borel-Cantelli lemma,

$$\limsup_{\mathbb{N}\ni k\to\infty}\frac{\log\max_{u\in\mathcal{N}_\delta(k)}\tau_u}{k}\le 4(1+10\delta)\quad\text{almost surely.}$$

We now turn to study $\{\tau_{\ell(s)};s\ge 0\}$.

On the one hand, for any $\delta>0$, we claim that almost surely for s large enough, the leftmost particle $\ell(s)$ at time s must have at least one descendant belonging to $\mathcal{N}_\delta(\lfloor s\rfloor+1)$.

In fact, let us write $\Upsilon_k:=\{\exists u\in\mathcal{N}(k+1):u\notin\mathcal{N}_\delta(k+1);\exists s\in[k,k+1],X_u(s)\le-\sqrt{2}s+\delta's\}$ with $\delta':=(\sqrt{2}-1)\delta/2$. By the many-to-one lemma, we get that for $k\ge 100/\delta$,

$$\mathbb{P}[\Upsilon_k]\le\frac{4}{\delta k\sqrt{2\pi}}e^{1+k-\frac{\delta^2k^2}{8}},$$

which is summable over k. It follows that

$$\mathbb{P}[\Upsilon_k\text{ infinitely often}]=0. \tag{3.4.5}$$

In view of (3.1.2), when s is large enough, $L(s)$ always lie below $-\sqrt{2}s+\delta's$ almost surely. Combining with (3.4.5), we obtain that almost surely for k sufficiently large,

$$\max_{s\in[k,k+1]}\tau_{\ell(s)}\le\max_{u\in\mathcal{N}_\delta(k+1)}\tau_u+1.$$

On the other hand, using similar arguments, one can say that almost surely for s sufficiently large, the leftmost located particle $\ell(s)$ at time s must come from one particle in $\mathscr{N}_\delta(\lfloor s \rfloor)$. This gives that almost surely for k sufficiently large,

$$\min_{s\in[k,k+1]} \tau_{\ell(s)} \geq \min_{u\in\mathscr{N}_\delta(k)} \tau_u - 1.$$

Thus we conclude that almost surely,

$$\lim_{s\to\infty} \frac{\log \tau_{\ell(s)}}{s} = 4. \qquad \square$$

3.5 Proof of Theorem 3.1.2

It suffices to show that almost surely $\lim_{\mathbb{N}\ni k\to\infty} \frac{\log \Theta_k}{k} = 2+2\sqrt{2}$, as the sequence $\{\Theta_s; s>0\}$ is monotone.

3.5.1 The lower bound of Theorem 3.1.2

This subsection is devoted to checking that : almost surely,

$$\liminf_{k\to\infty} \frac{\log \Theta_k}{k} \geq 2+2\sqrt{2}.$$

Proof. For $0 < a < \sqrt{2}$, we define

$$\mathscr{Z}_a(k) := \left\{u \in \mathscr{N}(k); X_u(k) \leq -ak\right\} \text{ and } Z_a(k) := \#\mathscr{Z}_a(k).$$

For $0 < \varepsilon < (1-\frac{a^2}{2})/2$ and $0 < \delta < 1$, we denote

$$\begin{aligned} E_k &:= \left\{Z_a(k) \geq \exp[k(1-\frac{a^2}{2}-\varepsilon)]\right\}, \\ D_k &:= \left\{\Theta_k \leq \exp[(2+2\sqrt{2}-\delta)k]\right\}. \end{aligned}$$

Let us estimate $\mathbb{P}[D_k \cap E_k]$.

For any $s > 0$ and $\beta > 0$, we write $\Gamma = \Gamma(s,\beta) := \left\{N(s) = 1, L(s) \leq -\beta s\right\}$. Similarly, let $\Gamma_u := \left\{N^u(s) = 1, L^u(s) \leq -\beta s\right\}$ for every $u \in \mathscr{N}(k)$. Then,

$$\mathbb{P}\Big[D_k \cap E_k\Big] \leq \mathbb{P}\Big[\Big(\bigcap_{u\in\mathscr{Z}_a(k)} \Gamma_u^c\Big) \cap E_k\Big] + \mathbb{P}\Big[\Big(\bigcup_{u\in\mathscr{Z}_a(k)} \Gamma_u\Big) \cap D_k\Big]. \qquad (3.5.1)$$

By the branching structure, we obtain that

$$\mathbb{P}\Big[\Big(\bigcap_{u\in\mathscr{Z}_a(k)}\Gamma_u^c\Big)\cap E_k\Big]\le\mathbb{P}\Big[\Big(1-\mathbb{P}[\Gamma]\Big)^{Z_a(k)};E_k\Big]\le e^{-\mathbb{P}[\Gamma]\exp[k(1-\frac{a^2}{2}-\varepsilon)]}. \tag{3.5.2}$$

Clearly, $\mathbb{P}[\Gamma]=e^{-s}\mathbb{P}[B_s\le-\beta s]$. By (3.2.5), one sees that, for $\varepsilon>0$ small and s large enough,

$$\mathbb{P}\Big[\Big(\bigcap_{u\in\mathscr{Z}_a(k)}\Gamma_u^c\Big)\cap E_k\Big]\le\exp\Big\{-\exp\Big[-s(1+\frac{\beta^2}{2}+\varepsilon)+k(1-\frac{a^2}{2}-\varepsilon)\Big]\Big\}, \tag{3.5.3}$$

which is bounded by $e^{-e^{k\varepsilon}}$ if we choose $s=\frac{1-\frac{a^2}{2}-2\varepsilon}{1+\frac{\beta^2}{2}+\varepsilon}k$ with k sufficiently large.

It remains to bound $\Omega:=\mathbb{P}\Big[\Big(\bigcup_{u\in\mathscr{Z}_a(k)}\Gamma_u\Big)\cap D_k\Big]$ for $s=\frac{1-\frac{a^2}{2}-2\varepsilon}{1+\frac{\beta^2}{2}+\varepsilon}k$. Recalling the definition of Θ_k, one sees that for any $\rho\in(0,2)$,

$$\begin{aligned}\Omega\le\ &\mathbb{P}\Big[\bigcup_{u\in\mathscr{Z}_a(k)}\Big\{\tau_u<e^{\rho k}\Big\}\Big]\\&+\mathbb{P}\Big[\Big(\bigcup_{u\in\mathscr{Z}_a(k)}\Gamma_u\Big)\cap\Big(\bigcap_{u\in\mathscr{Z}_a(k)}\Big\{e^{\rho k}\le\tau_u\le e^{(2+2\sqrt{2}-\delta)k}\Big\}\Big)\Big]=:\Omega_a+\Omega_b.\end{aligned} \tag{3.5.4}$$

We choose now $\rho=1-2\varepsilon$ and $z=(\sqrt{2}-\frac{a^2}{2\sqrt{2}}-\frac{\varepsilon}{\sqrt{2}})k$. Then comparing $T^u(z)$ and $e^{\rho k}$ for every $u\in\mathscr{Z}_a(k)$ tells us that

$$\Omega_a\le\mathbb{P}\Big[\bigcup_{u\in\mathscr{Z}_a(k)}\{T^u(z)<e^{\rho k}\}\Big]+\mathbb{P}\Big[\bigcup_{u\in\mathscr{Z}_a(k)}\{\tau_u<e^{\rho k}\le T^u(z)\}\Big]. \tag{3.5.5}$$

It follows from the branching property that the first term of the right-hand side is bounded by $\mathbb{E}[Z_a(k)]\mathbb{P}[T(z)<e^{\rho k}]$, which is $e^k\mathbb{P}[B_k\le-ak]\mathbb{P}[T(z)<e^{\rho k}]$ by the many-to-one lemma. In view of the inequalities (3.2.4) and (3.2.13), one immediately has

$$\mathbb{P}\Big[\bigcup_{u\in\mathscr{Z}_a(k)}\{T^u(z)<e^{\rho k}\}\Big]\le e^k\frac{\sqrt{k}}{ak}e^{-a^2k/2}c_5e^{\rho k}z^2e^{-\sqrt{2}z}\le e^{-\eta k}, \tag{3.5.6}$$

for some $\eta:=\eta(\varepsilon)>0$ small enough.

For the second term of the right-hand side in (3.5.5), we observe that for any $u\in\mathscr{Z}_a(k)$, $\{\tau_u<e^{\rho k}\le T^u(z)\}$ implies that at time $\tau_u<e^{\rho k}$, the rightmost position $R(k+\tau_u)$ is exactly equal to $R^u(\tau_u)+X_u(k)$, which is less than $m(\tau_u)+z-ak$. Hence, the event $\cup_{u\in\mathscr{Z}_a(k)}\{\tau_u<e^{\rho k}\le T^u(z)\}$

ensures that there exists some time $r < e^{\rho k}$ such that the rightmost position $R(k+r)$ is less than $m(r)+z-ak$. This gives that

$$\mathbb{P}\Big[\bigcup_{u\in\mathscr{Z}_a(k)}\{\tau_u < e^{\rho k} \le T^u(z)\}\Big] \le \mathbb{P}\Big[\exists r \le e^{\rho k} \text{ s.t. } R(k+r) \le m(r)+z-ak\Big]. \tag{3.5.7}$$

Notice that with our choice of ρ and z, Proposition 3.2.5 can be applied to show that for all k sufficiently large,

$$\mathbb{P}\Big[\bigcup_{u\in\mathscr{Z}_a(k)}\{\tau_u < e^{\rho k} \le T^u(z)\}\Big] \le e^{-\eta k}. \tag{3.5.8}$$

Combined with (3.5.6), the inequality (3.5.5) becomes $\Omega_a \le 2e^{-\eta k}$.

As shown in (3.5.4), it remains to study Ω_b. For the particles $u \in \mathscr{Z}_a(k)$ such that $N^u(s)=1$, we focus on the subtree rooted at u but started from time $k+s$. Define

$$\widetilde{R}^u(t) := \max\Big\{X_v(k+s+t)-X_u(k+s); v\in\mathscr{N}(k+s+t), u<v\Big\}, \forall t\ge 0;$$

and

$$\widetilde{T}^u(y) := \inf\Big\{t\ge 1; \widetilde{R}^u(t) \ge m(t)+y\Big\}, \forall y>1.$$

Since $(\widetilde{R}^u(t), t\ge 0)$ is distributed as $(R(t), t\ge 0)$, $\widetilde{T}^u(y)$ has the same law as $T(y)$. Let us take $\sqrt{2}x = k(2+2\sqrt{2}-\delta/2)$. Comparing $\widetilde{T}^u(x)$ with $e^{k(2+2\sqrt{2}-\delta)}$ yields that

$$\begin{aligned}
\Omega_b &\le \mathbb{P}\Big[\exists\, \omega\in\mathscr{Z}_a(k), \text{ s.t. } N^\omega(s)=1, L^\omega(s)\le -\beta s, \widetilde{T}^\omega(x) \le e^{k(2+2\sqrt{2}-\delta)}\Big] \\
&+ \mathbb{P}\Big[\exists u\in\mathscr{Z}_a(k) \text{ s.t. } N^u(s)=1, L^u(s)\le -\beta s, e^{(1-2\varepsilon)k} \le \tau_u \le e^{k(2+2\sqrt{2}-\delta)} < \widetilde{T}^u(x)\Big] \\
&=: \Omega_{b1}+\Omega_{b2}.
\end{aligned}$$

By first conditioning on $\mathscr{F}_{k+s}$ and then on $\mathscr{F}_k$, one has

$$\begin{aligned}
\Omega_{b1} &\le \mathbb{E}\Big[Z_a(k)\Big]\mathbb{P}\Big[\Gamma\Big]\mathbb{P}\Big[T(x)\le e^{k(2+2\sqrt{2}-\delta)}\Big] \\
&= e^k\mathbb{P}\Big[B_k \le -ak\Big]e^{-s}\mathbb{P}\Big[B_s\le -\beta s\Big]\mathbb{P}\Big[T(x)\le e^{k(2+2\sqrt{2}-\delta)}\Big].
\end{aligned} \tag{3.5.9}$$

For $0<\varepsilon<\min\{\delta/8, (1-\frac{a^2}{2})/2\}$, by (3.2.13) and (3.2.4),

$$\Omega_{b1} \le e^{3\varepsilon k}c_5x^2e^{-\sqrt{2}x}e^{k(2+2\sqrt{2}-\delta)} \le c_{11}k^2e^{-\varepsilon k}. \tag{3.5.10}$$

On the other hand, the event $\{\exists u \in \mathscr{Z}_a(k) \text{ s.t. } N^u(s) = 1, L^u(s) \le -\beta s, e^{(1-2\varepsilon)k} \le \tau_u \le e^{k(2+2\sqrt{2}-\delta)} < \widetilde{T}^u(x)\}$ implies that there exists some time $r \in [e^{(1-2\varepsilon)k} - s, e^{k(2+2\sqrt{2}-\delta)} - s]$ such that the rightmost position $R(k+s+r)$ is less than $-ak - \beta s + m(r) + x$. Thus,

$$\begin{aligned}\Omega_{b2} &= \mathbb{P}\Big[\exists u \in \mathscr{Z}_a(k) \text{ s.t. } N^u(s) = 1, L^u(s) \le -\beta s, e^{(1-2\varepsilon)k} \le \tau_u \le e^{k(2+2\sqrt{2}-\delta)} < \tilde{T}^u(x)\Big] \\ &\le \mathbb{P}\Big[\exists r \in [e^{(1-2\varepsilon)k}, e^{k(2+2\sqrt{2}-\delta)} + k], \text{ s.t. } R(r) \le m(r-k-s) + x - ak - \beta s\Big].\end{aligned}$$

By taking $a = \beta = 2 - \sqrt{2}$, we obtain $s = \frac{1-\frac{a^2}{2}-2\varepsilon}{1+\frac{\beta^2}{2}+\varepsilon}k = (\frac{1}{\sqrt{2}} - \varepsilon_1)k$ for some sufficiently small $\varepsilon_1 = \varepsilon_1(\varepsilon) > 0$. Let $\delta \ge 8\sqrt{2}\varepsilon_1$, then $m(r-k-s) + x - ak - \beta s \le m(r) - \varepsilon_1 k$. By Proposition 3.2.5, for k large enough,

$$\Omega_{b2} \le C_1 e^{-\varepsilon_1 k/6\sqrt{2}}. \tag{3.5.11}$$

Since $\Omega_b \le \Omega_{b1} + \Omega_{b2}$, it follows from (3.5.10) and (3.5.11) that $\Omega_b \le c_{11}k^2 e^{-\varepsilon k} + C_1 e^{-\varepsilon_1 k/6\sqrt{2}}$. Combined with the fact that $\Omega_a \le 2e^{-\eta k}$, (3.5.4) implies that

$$\Omega_1 \le \Omega_a + \Omega_b \le 2e^{-\eta k} + c_{11}k^2 e^{-\varepsilon k} + C_1 e^{-\varepsilon_1 k/6\sqrt{2}}.$$

According to the inequality (3.5.1), for $0 < \varepsilon < \delta/8$, $\eta(\varepsilon) > 0$, $0 < \varepsilon_1(\varepsilon) \le \delta/8\sqrt{2}$ with δ sufficiently small, and for all k sufficiently large,

$$\begin{aligned}\mathbb{P}[D_k \cap E_k] &\le \exp\Big[-e^{\left(k(1-\frac{a^2}{2}-\varepsilon)-s(1+\frac{\beta^2}{2}+\varepsilon)\right)}\Big] + \Omega_1 \\ &\le e^{-e^{\varepsilon k}} + 2e^{-\eta k} + c_{11}k^2 e^{-\varepsilon k} + C_1 e^{-\varepsilon_1 k/6\sqrt{2}}.\end{aligned}$$

Consequently,

$$\sum_k \mathbb{P}[D_k \cap E_k] < \infty.$$

By Borel-Cantelli lemma, $\mathbb{P}[D_k \cap E_k \text{ i.o.}] = 0$. Recall that $E_k := \Big\{Z_a(k) \ge \exp\Big[k(1-\frac{a^2}{2}-\varepsilon)\Big]\Big\}$. Biggins [35] showed that almost surely,

$$\lim_{k\to\infty} \frac{\log Z_a(k)}{k} = 1 - \frac{a^2}{2}. \tag{3.5.12}$$

Therefore, for any $\delta > 0$ small, $\liminf_{k\to\infty} \frac{\log \Theta_k}{k} \ge 2 + 2\sqrt{2} - \delta$ almost surely. This implies the lower bound of Theorem 3.1.2. $\square$

3.5.2 The upper bound of Theorem 3.1.2

It remains to prove the upper bound, namely, almost surely,

$$\limsup_{k\to\infty} \frac{\log \Theta_k}{k} \le 2 + 2\sqrt{2}. \tag{3.5.13}$$

Before bringing out the proof of (3.5.13), let us state some preliminary results first.

For $M \in \mathbb{N}_+$, define

$$\sigma_M := \inf\{s > 0; N(s) = M+1\}.$$

Clearly, σ_M is a stopping time with respect to $\{\mathscr{F}_s;\ s \ge 0\}$. Since $N(s)$ follows the geometric distribution, one sees that for any $s \ge 0$, $\mathbb{P}[\sigma_M \le s] = \mathbb{P}[N(s) \ge 1+M] = (1-e^{-s})^M$. Moreover, σ_M has a density function, denoted by f_M, as follows :

$$f_M(s) := 1_{(s\ge 0)} M e^{-s}(1-e^{-s})^{M-1} \le 1_{(s\ge 0)} M e^{-s}. \tag{3.5.14}$$

Recall that $L(s) = \inf_{u\in\mathscr{N}(s)} X_u(s)$. Let $L(\sigma_M)$ denote the leftmost position at time σ_M. Notice that at time σ_M, there are $M+1$ particles which occupy at most M different positions. This tells us that for any s, $\mu > 0$,

$$\mathbb{P}\Big[L(s) \le -\mu s \big| \sigma_M = s\Big] \le M\mathbb{P}\Big[B_s \le -\mu s\Big] \le \frac{M}{\mu s} e^{-\mu^2 s/2}, \tag{3.5.15}$$

where the last inequality holds because of (3.2.4).

Let $\varepsilon \in (0, 1/2)$. For $r > 1/\varepsilon$ and $0 < s < r$, we set $\lambda := \lambda(s,r) > 0$ such that $s(1+\frac{\lambda^2}{2}) = r$. Let

$$\begin{aligned}
\Phi(r,\lambda) &:= \{\sigma_M > r-1\} \cup \{\varepsilon r \le \sigma_M \le r-1, L(\sigma_M) \le -\lambda(\sigma_M, r)\sigma_M\}, &(3.5.16)\\
\Psi(r,\lambda) &:= \big\{\varepsilon r \le \sigma_M \le r-1, L(\sigma_M) \ge -\lambda(\sigma_M, r)\sigma_M\big\}. &(3.5.17)
\end{aligned}$$

We have the following lemma, which gives some results of the random vector $(\sigma_M, L(\sigma_M))$.

Lemma 3.5.1. **(i)** *There exists a constant $c_{12} > 0$ such that*

$$\mathbb{P}\Big[\Phi\Big(r,\lambda\Big)\Big] \le c_{12} M^2 r e^{-r}. \tag{3.5.18}$$

(ii) *There exists a constant $c_{13} > 0$ such that*

$$\mathbb{E}\left[e^{2\sigma_M - \sqrt{2}L(\sigma_M)}; \Psi(r,\lambda)\right] \le c_{13} M^2 r^2 e^{\sqrt{2}r}. \tag{3.5.19}$$

Proof of Lemma 3.5.1. (i) Observe that

$$\begin{aligned}
\mathbb{P}\Big[\Phi\Big(r,\lambda\Big)\Big] &\leq \mathbb{P}\Big[\sigma_M > r-1\Big] + \mathbb{P}\Big[\varepsilon r \leq \sigma_M \leq r-1, L(\sigma_M) \leq -\lambda(\sigma_M, r)\sigma_M\Big] \\
&= \int_{r-1}^{\infty} f_M(s)ds + \int_{\varepsilon r}^{r-1} \mathbb{P}\Big[L(s) \leq -\lambda(s,r)s \Big| \sigma_M = s\Big] f_M(s)ds \\
&\leq \int_{r-1}^{\infty} Me^{-s}ds + \int_{\varepsilon r}^{r-1} \frac{M}{\lambda(s,r)s} e^{-\lambda(s,r)^2 s/2} Me^{-s}ds,
\end{aligned}$$

where the last inequality follows from (3.5.14) and (3.5.15). A few lines of simple computation yield (3.5.18).

(ii) Let us prove the inequality (3.5.19). By Fubini's theorem, we rewrite the expectation $\mathbb{E}\Big[e^{2\sigma_M - \sqrt{2}L(\sigma_M)}; \Psi(r,\lambda)\Big]$ as follows :

$$\begin{aligned}
&\int_{\varepsilon r}^{r-1} e^{2s} f_M(s) \mathbb{E}\Big[e^{-\sqrt{2}L(s)}; L(s) \geq -\lambda(s,r)s \Big| \sigma_M = s\Big] ds \qquad (3.5.20)\\
= &\int_{\varepsilon r}^{r-1} e^{2s} f_M(s) \int_{-\lambda(s,r)s}^{+\infty} \sqrt{2}e^{-\sqrt{2}x} \mathbb{P}\Big[-\lambda(s,r)s \leq L(s) \leq x \Big| \sigma_M = s\Big] dxds.
\end{aligned}$$

For $\varepsilon r \leq s \leq (r-1)$, one sees that $\lambda(s,r) = \sqrt{2(r-s)/s} \geq \sqrt{2/(r-1)} > 0$. We choose $0 < \lambda_0 = \min\{1-\sqrt{2}/2, \sqrt{2/(r-1)}\}$ so that

$$\begin{aligned}
&\int_{-\lambda(s,r)s}^{+\infty} \sqrt{2}e^{-\sqrt{2}x} \mathbb{P}\Big[-\lambda(s,r)s \leq L(s) \leq x \Big| \sigma_M = s\Big] dx \\
\leq &\int_{-\lambda_0 s}^{+\infty} \sqrt{2}e^{-\sqrt{2}x} dx + \int_{-\lambda(s,r)s}^{-\lambda_0 s} \sqrt{2}e^{-\sqrt{2}x} \mathbb{P}\Big[-\lambda(s,r)s \leq L(s) \leq x \Big| \sigma_M = s\Big] dx \\
\leq &\; e^{(\sqrt{2}-1)s} + \int_{-\lambda(s,r)s}^{-\lambda_0 s} \sqrt{2}e^{-\sqrt{2}x} \mathbb{P}\Big[-\lambda(s,r)s \leq L(s) \leq x \Big| \sigma_M = s\Big] dx. \qquad (3.5.21)
\end{aligned}$$

The last term on the right-hand side of (3.5.21), by a change of variable $x = -\mu s$, becomes

$$\begin{aligned}
&\int_{\lambda_0}^{\lambda(s,r)} \sqrt{2}se^{\sqrt{2}\mu s} \mathbb{P}\Big[-\lambda(s,r)s \leq L(s) \leq -\mu s \Big| \sigma_M = s\Big] d\mu \\
\leq &\int_{\lambda_0}^{\lambda(s,r)} \sqrt{2}se^{\sqrt{2}\mu s} \mathbb{P}\Big[L(s) \leq -\mu s \Big| \sigma_M = s\Big] d\mu \\
\leq &\int_{\lambda_0}^{\lambda(s,r)} \sqrt{2}se^{\sqrt{2}\mu s} \frac{M}{\mu s} e^{-\mu^2 s/2} d\mu,
\end{aligned}$$

where the last inequality comes from (3.5.15). Going back to (3.5.20),

$$\mathbb{E}\Big[e^{2\sigma_M - \sqrt{2}L(\sigma_M)}; \Psi(r,\lambda)\Big] \leq \int_{\varepsilon r}^{r-1} e^{2s} f_M(s) \bigg(\int_{\lambda_0}^{\lambda(s,r)} \sqrt{2s}\frac{M}{\mu} e^{(\sqrt{2}\mu - \frac{\mu^2}{2})s} d\mu + e^{(\sqrt{2}-1)s}\bigg) ds.$$

By (3.5.14), this is bounded by

$$\int_{\varepsilon r}^{r-1}\int_{\lambda_0}^{\lambda(s,r)}\sqrt{2s}\frac{M^2}{\mu}\exp\Big\{(1+\sqrt{2}\mu-\frac{\mu^2}{2})s\Big\}d\mu ds+M\int_{\varepsilon r}^{r-1}e^{\sqrt{2}s}ds.$$

Notice that if $s\le r/2$, then $\lambda(s,r)=\sqrt{2(\frac{r}{s}-1)}\ge\sqrt{2}$. It follows that

$$\max_{0<\mu\le\lambda(s,r)}(1+\sqrt{2}\mu-\frac{\mu^2}{2})s=2s\le r<\sqrt{2}r. \tag{3.5.22}$$

Otherwise, $r/2<s<r$ implies $\lambda(s,r)<\sqrt{2}$. Hence, $\max_{0<\mu\le\lambda(s,r)}(1+\sqrt{2}\mu-\frac{\mu^2}{2})s$ is achieved when $\mu=\lambda(s,r)$, which equals

$$\Big(1+\sqrt{2}\lambda(s,r)-\frac{\lambda(s,r)^2}{2}\Big)s=\Big(\frac{1+\sqrt{2}\lambda(s,r)-\frac{\lambda(s,r)^2}{2}}{1+\frac{\lambda(s,r)^2}{2}}\Big)r. \tag{3.5.23}$$

It is bounded by $\sqrt{2}r$ since $\max_{z\ge 0}\frac{1+\sqrt{2}z-z^2/2}{1+z^2/2}=\sqrt{2}$. Combing the two cases, we obtain that

$$\max_{\varepsilon r\le s\le r-1;0<\mu\le\lambda(s,r)}e^{(1+\sqrt{2}\mu-\frac{\mu^2}{2})s}\le e^{\sqrt{2}r}.$$

This implies that

$$\begin{aligned}\mathbb{E}\Big[e^{2\sigma_M-\sqrt{2}L(\sigma_M)};\Psi(r,\lambda)\Big] &\le \int_{\varepsilon r}^{r-1}\int_{\lambda_0}^{\lambda(s,r)}\sqrt{2s}\frac{M^2}{\mu}e^{\sqrt{2}r}d\mu ds+Mre^{\sqrt{2}r}\\ &\le M^2e^{\sqrt{2}r}\int_{\varepsilon r}^{r-1}\sqrt{2s}\frac{\lambda(s,r)}{\lambda_0}ds+Mre^{\sqrt{2}r}.\end{aligned}$$

As $\lambda_0=\min\{1-\sqrt{2}/2,\ \sqrt{2/(r-1)}\}$, we deduce that $\mathbb{E}\Big[e^{2\sigma_M-\sqrt{2}L(\sigma_M)};\Psi(r,\lambda)\Big]\le c_{13}M^2r^2e^{\sqrt{2}r}$, which completes the proof of (ii) in Lemma 3.5.1. □

Let us turn to prove the upper bound of Θ_k.

Proof of (3.5.13). For any $u\in\mathscr{N}(k)$ and $t>0$, we denote $A_u(t):=\{\tau_u>t\}$. Then $\{\Theta_k>t\}=\cup_{u\in\mathscr{N}(k)}A_u(t)$.

For any $\theta\in\mathbb{Q}\cap(0,1)$, let

$$a_j:=\sqrt{2}-j\theta,\ \ b_j:=\sqrt{2}-(j-1)\theta,\ \text{for } j=1,\dots,K:=K(\theta)=\lfloor\frac{\sqrt{2}}{\theta}\rfloor,$$

so that $0<a_j<\sqrt{2}$ for all $j\le K$.

Let $I_k(a,b) := \{u \in \mathcal{N}(k); ak \le X_u(k) \le bk\}$ for $-\infty < a < b < \infty$.

Given the event $\Xi := \Big\{ -\sqrt{2}k \le L(k) \le R(k) \le \sqrt{2}k \Big\}$, we can write

$$\mathcal{N}(k) = I_k(-\theta, \sqrt{2}) \cup \Big(\bigcup_{1\le j\le K} I_k(-b_j, -a_j) \Big),$$

so that

$$\{\Theta_k > t\} = \bigcup_{u\in\mathcal{N}(k)} A_u(t) = \Big(\bigcup_{u\in I_k(-\theta,\sqrt{2})} A_u(t) \Big) \cup \Big(\bigcup_{1\le j\le K} \bigcup_{u\in I_k(-b_j,-a_j)} A_u(t) \Big).$$

As a consequence,

$$\mathbb{P}\Big[\Big\{\Theta_k > t\Big\} \cap \Xi\Big] \le \mathbb{P}\Big[\bigcup_{u\in I_k(-\theta,\sqrt{2})} A_u(t) \cap \Xi \Big] + \sum_{j=1}^{K} \mathbb{P}\Big[\bigcup_{u\in I_k(-b_j,-a_j)} A_u(t) \cap \Xi \Big]. \tag{3.5.24}$$

We first estimate $\mathbb{P}\Big[\bigcup_{u\in I_k(-\theta,\sqrt{2})} A_u(t) \cap \Xi\Big]$.

For any particle $u \in \mathcal{N}(k)$, let $\sigma_M^u := \inf\{s > 0; N^u(s) = 1+M\}$. Recall that $L^u(t) = \min\{X_v(t+k) - X_u(k); v \in \mathcal{N}(k+s), u < v\}$ for any $t > 0$. By the branching property, conditioned on $\mathscr{F}_k$, $\{\sigma_M^u, L^u(\sigma_M^u)\}_{u\in\mathcal{N}(k)}$ are i.i.d. copies of $(\sigma_M, L(\sigma_M))$.

Similarly, we define $\Phi^u(r,\lambda) := \{\sigma_M^u > r-1\} \cup \{\varepsilon r \le \sigma_M^u \le r-1, L^u(\sigma_M^u) \le -\lambda(r, \sigma_M^u)\sigma_M^u\}$ and $\Psi^u(r,\lambda) := \big\{\varepsilon r \le \sigma_M^u \le r-1, L^u(\sigma_M^u) \ge -\lambda(\sigma_M^u, r)\sigma_M^u\big\}$. One immediately observes that

$$\begin{aligned}\mathbb{P}\Big[\bigcup_{u\in I_k(-\theta,\sqrt{2})} A_u(t) \cap \Xi \Big] \le{}& \mathbb{P}\Big[\bigcup_{u\in I_k(-\theta,\sqrt{2})} \Phi^u(r,\lambda) \Big] \\ &+ \mathbb{P}\Big[\bigcup_{u\in I_k(-\theta,\sqrt{2})} A_u(t) \cap \Big(\Phi^u(r,\lambda)\Big)^c ; \Xi \Big].\end{aligned} \tag{3.5.25}$$

Conditioning on $\mathscr{F}_k$ yields that

$$\mathbb{P}\Big[\bigcup_{u\in I_k(-\theta,\sqrt{2})} \Phi^u(r,\lambda) \Big] \le \mathbb{E}\Big[\sum_{u\in I_k(-\theta,\sqrt{2})} 1_{\Phi^u(r,\lambda)} \Big] = \mathbb{E}\Big[\sum_{u\in I_k(-\theta,\sqrt{2})} 1 \Big] \mathbb{P}\Big[\Phi(r,\lambda)\Big]. \tag{3.5.26}$$

Clearly,

$$\mathbb{E}\Big[\sum_{u\in I_k(-\theta,\sqrt{2})} 1 \Big] \le \mathbb{E}[N(k)] = e^k. \tag{3.5.27}$$

It then follows from (3.5.18) that

$$\mathbb{P}\Big[\cup_{u\in I_k(-\theta,\sqrt{2})} \Phi^u(r,\lambda) \Big] \le c_{12} M^2 r e^{-r+k}. \tag{3.5.28}$$

We now choose $r = k(1+\varepsilon)$ and set $\Lambda_0 := \mathbb{P}\Big[\cup_{u\in I_k(-\theta,\sqrt{2})} A_u(t) \cap \Big(\Phi^u(r,\lambda)\Big)^c ; \Xi\Big]$. Then for all k large enough, (3.5.25) becomes

$$\mathbb{P}\Bigg[\bigcup_{u\in I_k(-\theta,\sqrt{2})} A_u(t) \cap \Xi\Bigg] \le c_{14} M^2 k e^{-\varepsilon k} + \Lambda_0. \tag{3.5.29}$$

It remains to estimate Λ_0. Since $\Big(\Phi^u(r,\lambda)\Big)^c \subset \{\sigma_M^u < \varepsilon r\} \cup \Psi^u(r,\lambda)$, we write

$$\Lambda_0 = \mathbb{P}\Big[\bigcup_{u\in I_k(-\theta,\sqrt{2})} A_u(t) \cap \Big(\Phi^u(r,\lambda)\Big)^c ; \Xi\Big] \le \Lambda_1 + \Lambda_2, \tag{3.5.30}$$

where

$$\begin{aligned} \Lambda_1 &:= \mathbb{P}\Bigg[\bigcup_{u\in I_k(-\theta,\sqrt{2})} A_u(t) \cap \{\sigma_M^u < \varepsilon r\} ; \Xi\Bigg] \\ \Lambda_2 &:= \mathbb{P}\Bigg[\bigcup_{u\in I_k(-\theta,\sqrt{2})} A_u(t) \cap \Psi^u(r,\lambda) ; \Xi\Bigg]. \end{aligned}$$

For any particle $u \in I_k(-\theta,\sqrt{2})$ such that $\{\sigma_M^u < \varepsilon r\}$, for any $y \ge 1$, we define

$$\widetilde{S}^u(y) := \min_{v\in\mathcal{N}^u(\varepsilon r)} T^v(y) + \varepsilon r. \tag{3.5.31}$$

Recall that $A_u(t) = \{\tau_u > t\}$. By comparing $\widetilde{S}^u(y)$ with t, we obtain that for any $t_1 \in (0,t)$,

$$\begin{aligned} \Lambda_1 &\le \mathbb{P}\Big[\bigcup_{u\in I_k(-\theta,\sqrt{2})} \Big\{t_1 \le \widetilde{S}^u(y) \le t < \tau_u\Big\} ; \Xi\Big] \qquad (3.5.32) \\ &+ \mathbb{P}\Big[\bigcup_{u\in I_k(-\theta,\sqrt{2})} \Big\{\widetilde{S}^u(y) > t ; \sigma_M^u < \varepsilon r\Big\}\Big] + \mathbb{P}\Big[\bigcup_{u\in I_k(-\theta,\sqrt{2})} \Big\{\widetilde{S}^u(y) < t_1\Big\}\Big] \\ &=: \Lambda_{1a} + \Lambda_{1b} + \Lambda_{1c}. \end{aligned}$$

For $\delta \in (0,1)$ and $\theta \in \mathbb{Q}\cap(0,1)$, we take $y = (\sqrt{2}+2)k(1+\theta)$, $t_1 = e^k$, $t = e^{\sqrt{2}y(1+2\delta)}$. As $\{\sigma_M^u < \varepsilon r\}$ implies $\{N^u(\varepsilon r) > M\}$, $\mathbb{P}[\widetilde{S}^u(y) > t ; \sigma_M^u < \varepsilon r | \mathscr{F}_{k+\varepsilon r}]$ is less than $\mathbb{P}[T(y) > t - \varepsilon r]^M$. Conditionally on $\mathscr{F}_{k+\varepsilon r}$, then by (3.5.27), we get

$$\Lambda_{1b} \le \mathbb{E}\Big[\sum_{u\in I_k(-\theta,\sqrt{2})} 1\Big] \mathbb{P}\Big[T(y) > t - \varepsilon r\Big]^M \le e^k \mathbb{P}\Big[T(y) > e^{\sqrt{2}y(1+\delta)}\Big]^M, \tag{3.5.33}$$

since $r = (1+\varepsilon)k$ with $\varepsilon \in (0,1/2)$. By (3.2.16), $\Lambda_{1b} \le e^k \times e^{-M\delta_1 y/3}$ for k large enough. We take $M = \frac{6}{\delta_1}$ to ensure that $\Lambda_{1b} \le e^{-k}$.

On the other hand, we observe that

$$\Lambda_{1c} \le \mathbb{E}\Big[\sum_{v\in\mathscr{N}(k+\varepsilon r)} 1_{\left(T^v(y)<e^k\right)}\Big] \le e^{k+\varepsilon r}\mathbb{P}\Big[T(y)<e^k\Big]. \tag{3.5.34}$$

By (3.2.13), $\Lambda_{1c} \le e^{k+\varepsilon r} c_5 y^2 e^{-\sqrt{2}y} e^k$. Thus $\Lambda_{1b}+\Lambda_{1c} \le 2e^{-k}$ for sufficiently large k.

Set $\Xi_1 := \Big\{R(k+\varepsilon r) \le 2(k+\varepsilon r)\Big\} \cap \Xi$. Then,

$$\Lambda_{1a} \le \mathbb{P}\Big[R(k+\varepsilon r) > 2(k+\varepsilon r)\Big] + \mathbb{E}\Big[\sum_{u\in I_k(-\theta,\sqrt{2})} 1_{(\tau_u>t\ge \widetilde{S}^u(y)\ge t_1)};\Xi_1\Big]. \tag{3.5.35}$$

We define $\Lambda_{1rest} := \mathbb{E}\Big[\sum_{u\in I_k(-\theta,\sqrt{2})} 1_{(\tau_u>t\ge \tilde{S}^u(y)\ge t_1)};\Xi_1\Big]$ for convenience. On the one hand, $\mathbb{P}[R(k+\varepsilon r) > 2(k+\varepsilon r)] \le e^{-k}$ because of (3.2.15). On the other hand, we have

$$\Lambda_{1rest} \;\le\; \mathbb{E}\Big[\sum_{u\in I_k(-\theta,\sqrt{2})} \int_{t_1}^{t} \mathbf{1}_{(\widetilde{S}^u(y)\in dr')}\mathbb{P}\Big[\tau_u > r';\Xi_1\Big|\mathscr{F}_k,\mathscr{F}^u_\infty\Big]\Big].$$

Since $\{\widetilde{S}^u(y)=r'\} \subset \{R^u(r') \ge L^u(\varepsilon r)+m(r'-\varepsilon r)+y\}$, the event $\{\tau_u>r'\}$ conditioned on $\{\widetilde{S}^u(y)=r'\}$ implies $\cup_{w\in\mathscr{N}(k)\setminus\{u\}}\Big\{X_w(k)+R^w(r') > X_u(k)+L^u(\varepsilon r)+m(r'-\varepsilon r)+y\Big\}$. Further, this set is contained in $\cup_{w\in\mathscr{N}(k)\setminus\{u\}}\Big\{R^w(r') > m(r')+X_u(k)+L^u(\varepsilon r)+y-\sqrt{2}\varepsilon r - X_w(k)\Big\}$. As Ξ_1 guarantees that $\Big(X_u(k)+L^u(\varepsilon r)+y-\sqrt{2}\varepsilon r-X_w(k)\Big)_+ \le C_2 k$, the inequality (3.2.2) can be applied to show that

$$\begin{aligned}
&\mathbb{E}\Big[\sum_{u\in I_k(-\theta,\sqrt{2})} \int_{t_1}^{t} \mathbf{1}_{(\widetilde{S}^u(y)\in dr')}\mathbb{P}\Big[\tau_u > r';\Xi_1\Big|\mathscr{F}_k,\mathscr{F}^u_\infty\Big]\Big]\\
\le\;& \mathbb{E}\Big[\sum_{u\in I_k(-\theta,\sqrt{2})} \int_{t_1}^{t} \mathbf{1}_{(\tilde{S}^u(y)\in dr')} \sum_{w\in\mathscr{N}(k)\setminus\{u\}} c_2(1+C_2k)^2 e^{-\sqrt{2}\left(X_u(k)+L^u(\varepsilon r)+y-\sqrt{2}\varepsilon r-X_w(k)\right)}\Big]\\
\le\;& c_{15}k^2 e^{-\sqrt{2}y+2\varepsilon r}\mathbb{E}\Big[\sum_{u\in I_k(-\theta,\sqrt{2})}\sum_{w\in\mathscr{N}(k)\setminus\{u\}} e^{\sqrt{2}(X_w(k)-X_u(k))}\Big]\mathbb{E}\Big[e^{-\sqrt{2}L(\varepsilon r)}\Big],
\end{aligned}$$

where the last inequality follows from the fact that for $u\in\mathscr{N}(k)$, $L^u(\varepsilon r)$ are independent of $\mathscr{F}_k$ and are independent copies of $L(\varepsilon r)$.

Whereas by the estimation of $\mathbb{E}\left[\sum_{u\in\mathscr{N}_\delta(k)}\sum_{w\in\mathscr{N}(k)\setminus\{u\}} e^{\sqrt{2}X_w(k)}\right]$ in Section 3.4,

$$\begin{aligned}
&\mathbb{E}\left[\sum_{u\in I_k(-\theta,\sqrt{2})}\sum_{w\in\mathscr{N}(k)\setminus\{u\}} e^{\sqrt{2}(X_w(k)-X_u(k))}\right]\\
= \ & 2\int_0^k e^{2k-s}ds\int_{\mathbb{R}}\mathbb{P}\left[B_s\in dx\right]\mathbb{E}\left[e^{-\sqrt{2}B_k};-\theta k\le B_k\le\sqrt{2}k\Big|B_s=x\right]\mathbb{E}\left[e^{\sqrt{2}B_k}\Big|B_s=x\right]\\
= \ & 2\int_0^k e^{3k-2s}\mathbb{E}\left[e^{\sqrt{2}B_s-\sqrt{2}B_k};-\theta k\le B_k\le\sqrt{2}k\right]ds.
\end{aligned}$$

Because $\mathbb{E}\left[e^{\sqrt{2}B_s-\sqrt{2}B_k};-\theta k\le B_k\le\sqrt{2}k\right]\le e^{\sqrt{2}\theta k}\mathbb{E}\left[e^{\sqrt{2}B_s}\right]=e^{\sqrt{2}\theta k+s}$, we obtain that

$$\mathbb{E}\left[\sum_{u\in I_k(-\theta,\sqrt{2})}\sum_{w\in\mathscr{N}(k)\setminus\{u\}} e^{\sqrt{2}(X_w(k)-X_u(k))}\right]\le 2e^{3k+\sqrt{2}\theta k}. \tag{3.5.36}$$

Besides, $\mathbb{E}\left[e^{-\sqrt{2}L(\varepsilon r)}\right]\le\mathbb{E}\left[\sum_{v\in\mathscr{N}(\varepsilon r)}e^{-\sqrt{2}X_v(\varepsilon r)}\right]=e^{2\varepsilon r}$. As a result,

$$\Lambda_{1rest}\le c_{15}k^2e^{-\sqrt{2}y+2\varepsilon r}\times 2e^{3k+\sqrt{2}\theta k+2\varepsilon r}.$$

Recall that $0<\theta<1$, $y=(\sqrt{2}+2)k(1+\theta)$ and that $r=k(1+\varepsilon)$ with k large enough so that $r>1/\varepsilon$. Hence, $\Lambda_{1rest}\le c_{15}k^2e^{(1-2\sqrt{2})k}$ for $\varepsilon\in(0,\frac{\theta}{3})$. Going back to (3.5.35), we get $\Lambda_{1a}\le e^{-k}+c_{15}k^2e^{(1-2\sqrt{2})k}$ for all k sufficiently large.

Consequently, (3.5.32) becomes

$$\Lambda_1\le e^{-k}+c_{15}k^2e^{(1-2\sqrt{2})k}+2e^{-k}\le c_{16}e^{-\varepsilon k}. \tag{3.5.37}$$

It remains to estimate $\Lambda_2=\mathbb{P}\left[\bigcup_{u\in I_k(-\theta,\sqrt{2})}A_u(t)\cap\Psi^u(r,\lambda);\Xi\right]$ where $\Psi^u(r,\lambda)=\left\{\varepsilon r\le\sigma_M^u\le r-1, L^u(\sigma_M^u)\ge-\lambda(\sigma_M^u,r)\sigma_M^u\right\}$. For any particle $u\in\mathscr{N}(k)$ satisfying $\Psi^u(r,\lambda)$, define

$$\widehat{S}^u(y):=\min_{v\in\mathscr{N}^u(\sigma_M^u)}T^v(y)+\sigma_M^u,\ \text{for any } y>0. \tag{3.5.38}$$

Comparing $\widehat{S}^u(y)$ with t yields that

$$\begin{aligned}
\Lambda_2\le\ &\mathbb{P}\left[\bigcup_{u\in I_k(-\theta,\sqrt{2})}\left\{t_1\le\widehat{S}^u(y)\le t<\tau_u\right\}\cap\Psi^u(r,\lambda);\Xi\right]\\
&+\mathbb{P}\left[\bigcup_{u\in I_k(-\theta,\sqrt{2})}\left\{\widehat{S}^u(y)>t\right\}\cap\Psi^u(r,\lambda)\right]+\mathbb{P}\left[\bigcup_{u\in I_k(-\theta,\sqrt{2})}\left\{\widehat{S}^u(y)<t_1\right\}\cap\Psi^u(r,\lambda)\right]\\
&=:\Lambda_{2a}+\Lambda_{2b}+\Lambda_{2c}.
\end{aligned} \tag{3.5.39}$$

According to the definition of $\widehat{S}^u(y)$, one sees that $\mathbb{P}[\widehat{S}^u(y)>t;\ \Psi^u(r,\lambda)]\le\mathbb{P}[T(y)>t-r]^M$ and that $\mathbb{P}[\widehat{S}^u(y)<t_1;\ \Psi^u(r,\lambda)]\le 2M\mathbb{P}[T(y)<t_1]$. Recall that $r=(1+\varepsilon)k$, $M=6/\delta_1$, $y=(\sqrt{2}+2)k(1+\theta)$, $t_1=e^k$ and $t=e^{\sqrt{2}y(1+2\delta)}$. For any $-\infty<a<b<\infty$, by (3.2.16),

$$\mathbb{E}\left[\sum_{u\in I_k(a,b)}1_{\{\widehat{S}^u(y)>t\}\cap\Psi^u(r,\lambda)}\right]\le e^k\times\mathbb{P}[T(y)>e^{\sqrt{2}y(1+\delta)}]^M\le e^{-k}. \tag{3.5.40}$$

Meanwhile, by (3.2.13),

$$\mathbb{E}\left[\sum_{u\in I_k(a,b)}1_{\{\widehat{S}^u(y)<t_1\}\cap\Psi^u(r,\lambda)}\right]\le e^k\times 2M\mathbb{P}[T(y)<t_1]\le 2c_5Me^{-2k}. \tag{3.5.41}$$

Hence, taking $a=-\theta$ and $b=\sqrt{2}$ implies that $\Lambda_{2b}+\Lambda_{2c}\le e^{-k}+2c_5Me^{-2k}$. Let Ξ_2 be the event $\{\max_{0\le r_0\le r}R(k+r_0)\le 6k\}\cap\Xi$. We get

$$\Lambda_{2a}\le\mathbb{P}\Big[\max_{0\le r_0\le r}R(k+r_0)>6k\Big]+\mathbb{E}\left[\sum_{u\in I_k(-\theta,\sqrt{2})}1_{\{t_1\le\widehat{S}^u(y)\le t<\tau_u\}\cap\Psi^u(r,\lambda)};\ \Xi_2\right]. \tag{3.5.42}$$

By the many-to-one lemma, for k large enough,

$$\mathbb{P}\Big[\max_{0\le r_0\le r}R(k+r_0)>6k\Big]\le e^{-k}. \tag{3.5.43}$$

For the second term on the right-hand side of (3.5.42), we need to recount the arguments to estimate Λ_{1rest}. Let

$$\Lambda_{2rest}:=\mathbb{E}\left[\sum_{u\in I_k(-\theta,\sqrt{2})}1_{\{t_1\le\widehat{S}^u(y)\le t<\tau_u\}\cap\Psi^u(r,\lambda)};\ \Xi_2\right]. \tag{3.5.44}$$

It immediately follows that

$$\Lambda_{2rest}\le\mathbb{E}\left[\sum_{u\in I_k(-\theta,\sqrt{2})}\int_{t_1}^{t}1_{(\widehat{S}^u(y)\in dr')}1_{\Psi^u(r,\lambda)}\times\mathbb{P}\Big[\{\tau_u>r'\}\cap\Xi_2\Big|\mathscr{F}_k,\mathscr{F}^u_\infty\Big]\right]. \tag{3.5.45}$$

Comparing τ_u with $\widehat{S}^u(y)$ tells that

$$\begin{aligned}\Lambda_{2rest}\le\mathbb{E}\Bigg[&\sum_{u\in I_k(-\theta,\sqrt{2})}\int_{t_1}^{t}1_{\left(\widehat{S}^u(y)\in dr'\right)}1_{\Psi^u(r,\lambda)}\times\\&\mathbb{E}\Bigg[1_{\Xi_2}\sum_{w\in\mathscr{N}(k)\setminus\{u\}}1_{\left(R^w(r')>m(r')-\sqrt{2}\sigma^u_M+y+X_u(k)-X_w(k)+L^u(\sigma^u_M)\right)}\Bigg|\mathscr{F}_k,\mathscr{F}^u_\infty\Bigg]\Bigg].\end{aligned}$$

On the event $\Xi_2 \cap \Psi^u(r,\lambda)$, we have $1+\Big(-\sqrt{2}\sigma_M^u + y + X_u(k) - X_w(k) + L^u(\sigma_M^u)\Big)_+ \leq C_3 k$. Applying the inequality (3.2.2) for $R^w(r')$ yields that

$$\begin{aligned}\Lambda_{2rest} &\leq \mathbb{E}\left[\sum_{u\in I_k(-\theta,\sqrt{2})} 1_{\Psi^u(r,\lambda)} \times \sum_{w\in\mathscr{N}(k)\setminus\{u\}} C_3^2 k^2 e^{-\sqrt{2}y} e^{\sqrt{2}(X_w(k)-X_u(k))} e^{2\sigma_M^u - \sqrt{2}L^u(\sigma_M^u)}\right] \\ &= C_3^2 k^2 e^{-\sqrt{2}y} \mathbb{E}\left[\sum_{u\in I_k(-\theta,\sqrt{2})} \sum_{w\in\mathscr{N}(k)\setminus\{u\}} e^{\sqrt{2}(X_w(k)-X_u(k))}\right] \mathbb{E}\left[e^{2\sigma_M - \sqrt{2}L(\sigma_M)}; \Psi(r,\lambda)\right],\end{aligned}$$

by the fact that $\{\sigma_M^u, L^u(\sigma_M^u)\}$ are i.i.d. and independent of $\mathscr{F}_k$. Recall that $r=(1+\varepsilon)k$ with $\varepsilon\in(0,\theta/3)$. It then follows from (3.5.36) and (3.5.19) that

$$\Lambda_{2rest} \leq C_3^2 k^2 e^{-(2+2\sqrt{2})k(1+\theta)} \times 2e^{3k+\sqrt{2}\theta k} \times c_{13} M^2 r^2 e^{\sqrt{2}r} \leq c_{17} k^4 M^2 e^{-(\sqrt{2}-1)k}. \tag{3.5.46}$$

Consequently, $\Lambda_{2a} \leq e^{-k} + c_{17} k^4 M^2 e^{-(\sqrt{2}-1)k}$.

Therefore, for k sufficiently large, the inequality (3.5.39) becomes

$$\Lambda_2 \leq e^{-k} + 2c_5 M e^{-2k} + e^{-k} + c_{17} k^4 M^2 e^{-(\sqrt{2}-1)k} \leq c_{18} M^2 e^{-\varepsilon k}. \tag{3.5.47}$$

Combined with (3.5.37), $\Lambda_0 \leq c_{16} e^{-\varepsilon k} + c_{18} M^2 e^{-\varepsilon k}$. Going back to (3.5.29), we conclude that

$$\mathbb{P}\left[\bigcup_{u\in I_k(-\theta,\sqrt{2})} A_u(t)\cap\Xi\right] \leq c_{19} M^2 e^{-\varepsilon k/2}. \tag{3.5.48}$$

To complete the proof, we still need to evaluate $\mathbb{P}\Big[\cup_{u\in I_k(-b_j,-a_j)} A_u(t)\cap\Xi\Big]$. Recall that for any particle $u\in\mathscr{N}(k)$, $\sigma_M^u = \inf\{s>0; N^u(s)=1+M\}$ and

$$\Phi^u(r,\lambda) = \{\sigma_M^u > r-1\} \cup \{\varepsilon r \leq \sigma_M^u \leq r-1, L^u(\sigma_M^u) \leq -\lambda(\sigma_M^u, r)\sigma_M^u\},$$

for any $r>1/\varepsilon$ and $\lambda(s,r)=\sqrt{2(\frac{r}{s}-1)}$ with $0<s<r$. Clearly,

$$\mathbb{P}\left[\bigcup_{u\in I_k(-b_j,-a_j)} A_u(t);\Xi\right] \leq \mathbb{P}\left[\bigcup_{u\in I_k(-b_j,-a_j)} \Phi^u(r,\lambda)\right] + \mathbb{P}\left[\bigcup_{u\in I_k(-b_j,-a_j)} A_u(t)\cap\Big(\Phi^u(r,\lambda)\Big)^c;\Xi\right].$$

On the one hand,

$$\mathbb{P}\left[\bigcup_{u\in I_k(-b_j,-a_j)} \Phi^u(r,\lambda)\right] \leq \mathbb{E}\left[\sum_{u\in I_k(-b_j,-a_j)} 1\right] \mathbb{P}\Big[\Phi(r,\lambda)\Big]. \tag{3.5.49}$$

We now take $r = k(1-\frac{a_j^2}{2})(1+\varepsilon)$ with $\varepsilon > 0$ small so that $r \le 2k$. Recall that $a_j = \sqrt{2} - j\theta$ for $1 \le j \le \lfloor \frac{\sqrt{2}}{\theta} \rfloor$, with $\theta \in \mathbb{Q} \cap (0,1)$. Then note that each a_j is strictly positive. Thus, by the many-to-one lemma and (3.5.18), (3.5.49) becomes that

$$\begin{aligned}
\mathbb{P}\left[\bigcup_{u\in I_k(-b_j,-a_j)} \Phi^u(r,\lambda)\right] &\le e^k \mathbb{P}[-b_j k \le B_k \le -a_j k] \times c_{12} M^2 r e^{-r} \\
&\le c_{12} M^2 r e^{-r+k} \mathbb{P}[B_k \le -a_j k] \\
&\le c_{12} M^2 (2k) e^{-k(1-\frac{a_j^2}{2})(1+\varepsilon)+k} \left(\frac{\sqrt{k}}{a_j k} e^{-\frac{a_j^2}{2}k}\right),
\end{aligned}$$

where the last inequality follows from (3.2.4). $\mathbb{P}\left[\bigcup_{u\in I_k(-b_j,-a_j)} \Phi^u(r,\lambda)\right]$ is hence bounded by $c_{12}\frac{2M^2}{a_j}\sqrt{k} e^{-\sqrt{2}\theta\varepsilon k/2}$.

On the other hand, recalling that $\Psi^u(r,\lambda) = \{\varepsilon r \le \sigma_M^u \le r-1, L^u(\sigma_M^u) \ge -\lambda(\sigma_M^u, r)\sigma_M^u\}$, we deduce that

$$\mathbb{P}\left[\bigcup_{u\in I_k(-b_j,-a_j)} A_u(t) \cap \Big(\Phi^u(r,\lambda)\Big)^c; \Xi\right] \le \Lambda_1' + \Lambda_2', \tag{3.5.50}$$

where

$$\begin{aligned}
\Lambda_1' &:= \mathbb{P}\left[\bigcup_{u\in I_k(-b_j,-a_j)} A_u(t) \cap \Big\{\sigma_M^u < \varepsilon r\Big\}; \Xi\right], \\
\Lambda_2' &:= \mathbb{P}\left[\bigcup_{u\in I_k(-b_j,-a_j)} A_u(t) \cap \Psi^u(r,\lambda); \Xi\right].
\end{aligned}$$

Furthermore, by an argument similar to the one used in estimating Λ_1, we have $\Lambda_1' \le c_{20} e^{-\varepsilon k}$. Thus,

$$\mathbb{P}\left[\bigcup_{u\in I_k(-b_j,-a_j)} A_u(t); \Xi\right] \le c_{12}\frac{2M^2}{a_j}\sqrt{k} e^{-\sqrt{2}\theta\varepsilon k/2} + c_{20} e^{-\varepsilon k} + \Lambda_2'. \tag{3.5.51}$$

It remains to bound Λ_2'. Recall that $\widehat{S}^u(y) = \min_{v\in\mathcal{N}^u(\sigma_M^u)} T^v(y) + \sigma_M^u$ with $y = (\sqrt{2}+2)k(1+\theta)$. We observe that

$$\begin{aligned}
\Lambda_2' \le\ & \mathbb{P}\left[\bigcup_{u\in I_k(-b_j,-a_j)} \{\widehat{S}^u(y) > t\} \cap \Psi^u(r,\lambda)\right] + \mathbb{P}\left[\bigcup_{u\in I_k(-b_j,-a_j)} \{\widehat{S}^u(y) < t_1\} \cap \Psi^u(r,\lambda)\right] \\
&+ \mathbb{P}\left[\max_{0\le r_0\le r} R(k+r_0) > 6k\right] + \mathbb{P}\left[\bigcup_{u\in I_k(-b_j,-a_j)} \Big\{t_1 \le \widehat{S}^u(y) \le t < \tau_u\Big\} \cap \Psi^u(r,\lambda); \Xi_2\right].
\end{aligned}$$

In view of (3.5.40), (3.5.41) and (3.5.43),

$$\Lambda_2' \leq e^{-k} + 2c_5 M e^{-2k} + e^{-k} + \mathbb{E}\Big[\sum_{u\in I_k(-b_j,-a_j)} 1_{\{t_1\leq \widehat{S}^u(y)\leq t<\tau_u\}\cap\Psi^u(r,\lambda)};\Xi_2\Big]. \tag{3.5.52}$$

We define $\Lambda_{2rest}' := \mathbb{E}\Big[\sum_{u\in I_k(-b_j,-a_j)} 1_{\{t_1\leq \widehat{S}^u(y)\leq t<\tau_u\}\cap\Psi^u(r,\lambda)};\ \Xi_2\Big]$. Thus applying the analogous arguments to the estimation of Λ_{2rest} gives that

$$\Lambda_{2rest}' \leq c_{21}k^2 e^{-\sqrt{2}y}\mathbb{E}\Big[\sum_{u\in I_k(-b_j,-a_j)}\sum_{w\in\mathscr{N}(k)\setminus\{u\}} e^{\sqrt{2}(X_w(k)-X_u(k))}\Big] \times \mathbb{E}\Big[e^{2\sigma_M-\sqrt{2}L(\sigma_M)};\Psi^u(r,\lambda)\Big]. \tag{3.5.53}$$

Once again, by means of integrating with respect to the last time at which the most recent common ancestor of u and v was alive, $\mathbb{E}\Big[\sum_{u\in I_k(-b_j,-a_j)}\sum_{w\in\mathscr{N}(k)\setminus\{u\}} e^{\sqrt{2}(X_w(k)-X_u(k))}\Big]$ equals

$$\begin{aligned}
& 2\int_0^k e^{2k-s}ds\int_{\mathbb{R}}\mathbb{P}\Big[B_s\in dx\Big]\mathbb{E}\Big[e^{-\sqrt{2}B_k};-b_jk\leq B_k\leq -a_jk\Big|B_s=x\Big]\mathbb{E}\Big[e^{\sqrt{2}B_k}\Big|B_s=x\Big]\\
= & 2\int_0^k e^{3k-2s}\mathbb{E}\Big[e^{\sqrt{2}B_s-\sqrt{2}B_k};-b_jk\leq B_k\leq -a_jk\Big]ds\\
= & 2\int_0^k e^{3k-2s}ds\int_{-b_jk}^{-a_jk} e^{-\sqrt{2}x}\mathbb{P}\Big[B_k\in dx\Big]\mathbb{E}\Big[e^{\sqrt{2}B_s}\Big|B_k=x\Big].
\end{aligned}$$

Let $(b_s(x);0\leq s\leq k)$ denote a Brownian bridge from 0 to x of length k. Then $\mathbb{E}\Big[e^{\sqrt{2}B_s}\Big|B_k=x\Big]$ equals $\mathbb{E}\Big[e^{\sqrt{2}b_s(x)}\Big]$, which turns out to be $\exp(s(k-s+\sqrt{2}x)/k)$. Note that $a_j>0$ and that $b_j=a_j+\theta$. This gives that

$$\begin{aligned}
& \mathbb{E}\Big[\sum_{u\in I_k(-b_j,-a_j)}\sum_{w\in\mathscr{N}(k)\setminus\{u\}} e^{\sqrt{2}(X_w(k)-X_u(k))}\Big]\\
= & 2\int_0^k e^{3k-2s+s(k-s)/k}\int_{-b_jk}^{-a_jk}\frac{1}{\sqrt{2\pi k}}\exp\Big(-\frac{x^2}{2k}-\sqrt{2}x+\sqrt{2}\frac{s}{k}x\Big)dxds\\
\leq & 2\int_0^k e^{3k-2s+s(k-s)/k}\frac{\theta k}{\sqrt{2\pi k}}\exp\Big(-\frac{a_j^2}{2}k+\sqrt{2}b_jk-\sqrt{2}a_js\Big)ds,
\end{aligned}$$

which is bounded by $\sqrt{k}\exp\Big(3k-\frac{a_j^2}{2}k+\sqrt{2}b_jk\Big)$ since $\int_0^k\exp\Big(-2s+s(k-s)/k-\sqrt{2}a_js\Big)ds$ is less than 1. One hence sees that $\mathbb{E}\Big[\sum_{u\in I_k(-b_j,-a_j)}\sum_{w\in\mathscr{N}(k)\setminus\{u\}} e^{\sqrt{2}(X_w(k)-X_u(k))}\Big]$ is bounded by

$\sqrt{k}e^{2k+(1-\frac{a_j^2}{2}+\sqrt{2}a_j)k+\sqrt{2}\theta k}$. Going back to (3.5.53) and applying (3.5.19),

$$\begin{aligned}\Lambda'_{2rest} &\leq c_{21}k^2e^{-(2+2\sqrt{2})k(1+\theta)}\sqrt{k}e^{2k+(1-\frac{a_j^2}{2}+\sqrt{2}a_j)k+\sqrt{2}\theta k}c_{13}M^2r^2e^{\sqrt{2}r}\\ &\leq c_{22}k^{9/2}M^2e^{-2\theta k}\exp\left\{k\left[(\sqrt{2}+1)(1-a_j^2/2)+\sqrt{2}a_j-2\sqrt{2}\right]\right\},\end{aligned}$$

as $r=k(1-\frac{a_j^2}{2})(1+\varepsilon)$ with $\varepsilon\in(0,\frac{\theta}{3})$. Observe that

$$(\sqrt{2}+1)(1-a_j^2/2)+\sqrt{2}a_j-2\sqrt{2}=-\frac{\sqrt{2}+1}{2}\Big(a_j-(2-\sqrt{2})\Big)^2\leq 0. \tag{3.5.54}$$

We get $\Lambda'_{2rest}\leq c_{22}k^{9/2}M^2e^{-2\theta k}$, and thus for all k sufficiently large,

$$\Lambda'_2\leq c_{23}M^2e^{-\theta k}. \tag{3.5.55}$$

Consequently, by (3.5.51),

$$\mathbb{P}\left[\cup_{u\in I_k(-b_j,-a_j)}A_u(t);\Xi\right]\leq c_{12}\frac{2M^2}{a_j}\sqrt{k}e^{-\sqrt{2}\theta\varepsilon k/2}+c_{20}e^{-\varepsilon k}+c_{23}M^2e^{-\theta k}.$$

Summing over $j\in\{1,\cdots,K=\lfloor\frac{\sqrt{2}}{\theta}\rfloor\}$ implies that

$$\sum_{j=1}^{K}\mathbb{P}\left[\cup_{u\in I_k(-b_j,-a_j)}A_u(t);\Xi\right]\leq C(\theta)M^2e^{-\varepsilon\theta k/2}, \tag{3.5.56}$$

where $C(\theta)$ is a positive constant associated with θ (but independent of k, δ and M) and k is large enough.

Going back to (3.5.24), we combine (3.5.48) and (3.5.56) to say that

$$\begin{aligned}\mathbb{P}\left[\Big\{\tau(k)>t\Big\}\cap\Big\{-\sqrt{2}k\leq L(k)\leq R(k)\leq\sqrt{2}k\Big\}\right] &\leq c_{19}M^2e^{-\varepsilon k/2}+C(\theta)M^2e^{-\varepsilon\theta k/2}\\ &\leq \frac{C_1(\theta)}{\delta^2}e^{-\varepsilon\theta k/2},\end{aligned}$$

where $\theta\in\mathbb{Q}\cap(0,1)$, $\varepsilon\in(0,\frac{\theta}{3})$, $\delta>0$ and $t=\exp[k(2+2\sqrt{2})(1+\theta)(1+2\delta)]$ and k is sufficiently large.

According to the Borel-Cantelli Lemma, we conclude that for any $\theta\in\mathbb{Q}\cap(0,1)$ and any $\delta>0$,

$$\limsup_{k\to\infty}\frac{\log\Theta_k}{k}\leq(2+2\sqrt{2})(1+\theta)(1+2\delta),\qquad\text{almost surely.}$$

This implies the upper bound in Theorem 3.1.2. $\square$

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Chapitre 4

Scaling limit of the path leading to the leftmost particle in a branching random walk

The results in this chapter are contained in [56]

Summary. We consider a discrete-time branching random walk defined on the real line, which is assumed to be supercritical and in the boundary case. It is known that its leftmost position of the n-th generation behaves asymptotically like $\frac{3}{2}\log n$, provided the non-extinction of the system. The main goal of this paper, is to prove that the path from the root to the leftmost particle, after a suitable normalizatoin, converges weakly to a Brownian excursion.

Keywords. Branching random walk ; spinal decomposition.

4.1 Introduction

We consider a branching random walk, which is constructed according to a point process $\mathscr{L}$ on the line. Precisely speaking, the system is started with one initial particle at the origin. This particle is called the root, denoted by $\varnothing$. At time 1, the root dies and gives birth to some new particles, which form the first generation. Their positions constitute a point process distributed as $\mathscr{L}$. At time 2, each of these particles dies and gives birth to new particles whose positions – relative to that of their parent – constitute a new independent copy of $\mathscr{L}$. The system grows according to the same mechanism.

We denote by $\mathbb{T}$ the genealogical tree of the system, which is clearly a Galton-Watson tree rooted at $\varnothing$. If a vertex $u \in \mathbb{T}$ is in the n-th generation, we write $|u| = n$ and denote its position by $V(u)$. Then $\{V(u),\ |u| = 1\}$ follows the same law as $\mathscr{L}$. The family of positions $(V(u);\ u \in \mathbb{T})$ is viewed as our branching random walk.

Throughout the paper, the branching random walk is assumed to be in the boundary case (Biggins and Kyprianou [40]) :

$$\mathbb{E}\Big[\sum_{|u|=1} 1\Big] > 1, \qquad \mathbb{E}\Big[\sum_{|x|=1} \mathrm{e}^{-V(x)}\Big] = 1, \qquad \mathbb{E}\Big[\sum_{|x|=1} V(x)\mathrm{e}^{-V(x)}\Big] = 0. \tag{4.1.1}$$

For any $y \in \mathbb{R}$, let $y_+ := \max\{y, 0\}$ and $\log_+ y := \log(\max\{y, 1\})$. We also assume the following integrability conditions :

$$\mathbb{E}\Big[\sum_{|u|=1} V(u)^2\mathrm{e}^{-V(u)}\Big] \ < \ \infty, \tag{4.1.2}$$

$$\mathbb{E}[X(\log_+ X)^2] \ < \ \infty, \qquad \mathbb{E}[\widetilde{X}\log_+ \widetilde{X}] < \infty, \tag{4.1.3}$$

where

$$X := \sum_{|u|=1} \mathrm{e}^{-V(u)}, \qquad \widetilde{X} := \sum_{|u|=1} V(u)_+\mathrm{e}^{-V(u)}.$$

We define I_n to be the leftmost position in the n-th generation, i.e.

$$I_n := \inf\{V(u),\ |u| = n\}, \tag{4.1.4}$$

with $\inf\emptyset := \infty$. If $I_n < \infty$, we choose a vertex uniformly in the set $\{u : |u| = n,\ V(u) = I_n\}$ of leftmost particles at time n and denote it by $m^{(n)}$. We let $[\![\varnothing, m^{(n)}]\!] = \{\varnothing =: m_0^{(n)}, m_1^{(n)}, \ldots, m_n^{(n)} := m^{(n)}\}$ be the shortest path in $\mathbb{T}$ relating the root $\varnothing$ to $m^{(n)}$, and introduce the path from the root to $m^{(n)}$ as follows

$$(I_n(k);\ 0 \le k \le n) := (V(m_k^{(n)});\ 0 \le k \le n).$$

In particular, $I_n(0) = 0$ and $I_n(n) = I_n$. Let σ be the positive real number such that σ^2 equals $\mathbb{E}\Big[\sum_{|u|=1} V(u)^2\mathrm{e}^{-V(u)}\Big]$. For any $t \in (0,\infty)$, let $D([0,t],\mathbb{R})$ be the space of the functions on $[0,t]$ which are right-continuous and have left-hand limits and we equip $D([0,t],\mathbb{R})$ with the Skorohod topology (see Chapter 3 of Billingsley [42]). Our main result is as follows.

Theorem 4.1.1. *The rescaled path* $(\frac{I_n(\lfloor sn \rfloor)}{\sigma\sqrt{n}};\ 0 \le s \le 1)$ *converges in law in* $D([0,1],\mathbb{R})$*, to a normalized Brownian excursion* $(e_s;\ 0 \le s \le 1)$.

Remark 4.1.2. *It has been proved in [2], [87] and [4] that I_n is around $\frac{3}{2}\log n$. In [6], the authors proved that, for the model of branching Brownian motion, the time reversed path followed by the leftmost particle converges in law to a certain stochastic process.*

Let us say a few words about the proof of Theorem 4.1.1. We first consider the path leading to $m^{(n)}$, by conditioning that its ending point I_n is located atypically below $\frac{3}{2}\log n - z$ with large z. Then we apply the well-known spinal decomposition to show that this path, conditioned to $\{I_n \leq \frac{3}{2}\log n - z\}$, behaves like a simple random walk staying positive but tied down at the end. Such a random walk, being rescaled, converges in law to the Brownian excursion (see [61]). We then prove our main result by removing the condition of I_n. The main strategy is borrowed from [4], but with appropriate refinements.

The rest of the paper is organized as follows. In Section 4.2, we recall the spinal decomposition by a change of measures, which implies the useful many-to-one lemma. We prove a conditioned version of Theorem 4.1.1 in Section 4.3. In Section 4.4, we remove the conditioning and prove the theorem.

Throughout the paper, we use $a_n \sim b_n$ $(n \to \infty)$ to denote $\lim_{n\to\infty} \frac{a_n}{b_n} = 1$; and let $(c_i)_{i\geq 0}$ denote finite and positive constants. We write $\mathbb{E}[f; A]$ for $\mathbb{E}[f 1_A]$. Moreover, $\sum_\emptyset := 0$ and $\prod_\emptyset := 1$.

4.2 Lyons' change of measures via additive martingale

4.2.1 Spinal decomposition

For any $a \in \mathbb{R}$, let $\mathbb{P}_a$ be the probability measure such that $\mathbb{P}_a((V(u),\, u \in \mathbb{T}) \in \cdot) = \mathbb{P}((V(u) + a,\, u \in \mathbb{T}) \in \cdot)$. The corresponding expectation is denoted by $\mathbb{E}_a$. Let $(\mathscr{F}_n,\, n \geq 0)$ be the natural filtration generated by the branching random walk and let $\mathscr{F}_\infty := \vee_{n\geq 0}\mathscr{F}_n$. We introduce the following random variables :

$$W_n := \sum_{|u|=n} e^{-V(u)}, \qquad n \geq 0. \tag{4.2.1}$$

It follows immediately from (4.1.1) that $(W_n,\, n \geq 0)$ is a non-negative martingale with respect to $(\mathscr{F}_n)$. It is usually referred as the additive martingale. We define a probability measure $\mathbb{Q}_a$ on $\mathscr{F}_\infty$ such that for any $n \geq 0$,

$$\left.\frac{d\mathbb{Q}_a}{d\mathbb{P}_a}\right|_{\mathscr{F}_n} := e^a W_n. \tag{4.2.2}$$

For convenience, we write $\mathbb{Q}$ for $\mathbb{Q}_0$.

Let us give the description of the branching random walk under $\mathbb{Q}_a$ in an intuitive way, which is known as the spinal decomposition. We introduce another point process $\widehat{\mathscr{L}}$ with Radon-Nikodym derivative $\sum_{x\in\mathscr{L}} e^{-x}$ with respect to the law of $\mathscr{L}$. Under $\mathbb{Q}_a$, the branching random walk evolves as follows. Initially, there is one particle w_0 located at $V(w_0) = a$. At each step n, particles at generation n die and give birth to new particles independently according to the law of $\mathscr{L}$, except for the particle w_n which generates its children according to the law of $\widehat{\mathscr{L}}$. The particle w_{n+1} is chosen proportionally to $e^{-V(u)}$ among the children u of w_n. We still call $\mathbb{T}$ the genealogical tree of the process, so that $(w_n)_{n\geq 0}$ is a ray in $\mathbb{T}$, which is called the spine. This change of probabilities was presented in various forms ; see, for example [109], [87] and [54].

It is convenient to use the following notation. For any $u \in \mathbb{T}\setminus\{\varnothing\}$, let $\overleftarrow{u}$ be the parent of u, and

$$\Delta V(u) := V(u) - V(\overleftarrow{u}).$$

Let $\Omega(u)$ be the set of brothers of u, i.e. $\Omega(u) := \{v \in \mathbb{T} : \overleftarrow{v} = \overleftarrow{u},\ v \neq u\}$. Let δ denote the Dirac measure. Then under $\mathbb{Q}_a$, $\sum_{|u|=1} \delta_{\Delta V(u)}$ follows the law of $\widehat{\mathscr{L}}$. Further, We recall the following proposition, from [87] and [109].

Proposition 4.2.1. *(1) For any $|u| = n$, we have*

$$\mathbb{Q}_a[w_n = u|\mathscr{F}_n] = \frac{e^{-V(u)}}{W_n}. \tag{4.2.3}$$

(2) Under $\mathbb{Q}_a$, the random variables $\Big(\sum_{v\in\Omega(w_n)} \delta_{\Delta V(v)},\ \Delta V(w_n)\Big)$, $n \geq 1$ are i.i.d..

As a consequence of this proposition, we get the many-to-one lemma as follows :

Lemma 4.2.2. *There exists a centered random walk $(S_n;\ n \geq 0)$ with $\mathbb{P}_a(S_0 = a) = 1$ such that for any $n \geq 1$ and any measurable function $g : \mathbb{R}^n \to [0,\infty)$, we have*

$$\mathbb{E}_a\Bigg[\sum_{|u|=n} g(V(u_1),\ldots,V(u_n))\Bigg] = \mathbb{E}_a[e^{S_n-a} g(S_1,\ldots,S_n)], \tag{4.2.4}$$

where we denote by $[\![\varnothing,\, u]\!] = \{\varnothing =: u_0, u_1 \ldots, u_{|u|} := u\}$ the ancestral line of u in $\mathbb{T}$.

Note that by (4.1.3), S_1 has the finite variance $\sigma^2 = \mathbb{E}[S_1^2] = \mathbb{E}[\sum_{|u|=1} V(u)^2 e^{-V(u)}]$.

4.2.2 Convergence in law for the one-dimensional random walk

Let us introduce some results about the centered random walk (S_n) with finite variance, which will be used later. For any $0 \leq m \leq n$, we define $\underline{S}_{[m,n]} := \min_{m\leq j\leq n} S_j$, and $\underline{S}_n = \underline{S}_{[0,n]}$. Let $(T_k, H_k;\ k \geq 0)$ be the strict descending ladder epochs and heights of $(S_n;\ n \geq 0)$, i.e., $T_0 = 0$,

$H_0 := S_0$ and for any $k \geq 1$, $T_k := \inf\{j > T_{k-1} : S_j < H_{k-1}\}$, $H_k := S_{T_k}$. We denote by $U(\mathrm{d}x)$ the corresponding renewal measure on $\mathbb{R}_+$, in other words, $U(\mathrm{d}x) = \sum_{k\geq 0}\mathbb{P}(-H_k \in \mathrm{d}x)$. Let $R(x) := U([0,x])$ for any $x \geq 0$. For the random walk $(-S_n)$, we define $\underline{S}^-_{[m,n]}$, $\underline{S}^-_n$ and $R_-(x)$ similarly. It is known (see [67] p. 360) that there exists $c_0 > 0$ such that

$$\lim_{x\to\infty}\frac{R(x)}{x} = c_0. \tag{4.2.5}$$

Moreover, it is shown in [99] that there exist C_+, $C_- > 0$ such that for any $a \geq 0$,

$$\mathbb{P}_a\Big(\underline{S}_n \geq 0\Big) \sim \frac{C_+}{\sqrt{n}}R(a); \tag{4.2.6}$$

$$\mathbb{P}_a\Big(\underline{S}^-_n \geq 0\Big) \sim \frac{C_-}{\sqrt{n}}R_-(a). \tag{4.2.7}$$

We also state the following inequalities (see Lemmas 2.2 and 2.4 in [9], respectively).

Fact 4.2.3. **(i)** *There exists a constant $c_1 > 0$ such that for any $b \geq a \geq 0$, $x \geq 0$ and $n \geq 1$,*

$$\mathbb{P}\big(\underline{S}_n \geq -x;\ S_n \in [a-x, b-x]\big) \leq c_1(1+x)(1+b-a)(1+b)n^{-3/2}. \tag{4.2.8}$$

(ii) *Let $0 < \lambda < 1$. There exists a constant $c_2 > 0$ such that for any $b \geq a \geq 0$, x, $y \geq 0$ and $n \geq 1$,*

$$\mathbb{P}_x(S_n \in [y+a, y+b], \underline{S}_n \geq 0, \underline{S}_{[\lambda n,n]} \geq y) \leq c_2(1+x)(1+b-a)(1+b)n^{-3/2}. \tag{4.2.9}$$

Before we give the next lemma, we recall the definition of lattice distribution (see [67], p. 138). The distribution of a random variable X_1 is lattice, if it is concentrated on a set of points $\alpha + \beta\mathbb{Z}$, with α arbitrary. The largest β satisfying this property is called the span of X_1. Otherwise, the distribution of X_1 is called non-lattice. Recall also that for $t > 0$, $D([0,t],\mathbb{R}) = \{f : [0,t] \to \mathbb{R} : \forall s \in [0,t), f(s) = \lim_{r\downarrow s} f(r);$ and $\forall s \in (0,t], \lim_{r\uparrow s} f(r)$ exists$\}$, equipped with the Skorohod topology.

Lemma 4.2.4. *Let $(r_n)_{n\geq 0}$ be a sequence of real numbers such that $\lim_{n\to\infty}\frac{r_n}{\sqrt{n}} = 0$. Let $f : \mathbb{R}_+ \to \mathbb{R}$ be a Riemann integrable function. We suppose that there exists a non-increasing function $\overline{f} : \mathbb{R}_+ \to \mathbb{R}$ such that $|f(x)| \leq \overline{f}(x)$ for any $x \geq 0$ and $\int_{x\geq 0} x\overline{f}(x)dx < \infty$. For $0 < \Delta < 1$, let $F : D([0,\Delta], \mathbb{R}) \to [0,1]$ be continuous. Let $a \geq 0$.*

(I) Non-lattice case. *If the distribution of $(S_1 - S_0)$ is non-lattice, then there exists a constant $C_1 > 0$ such that*

$$\lim_{n\to\infty} n^{3/2}\mathbb{E}\Big[F\Big(\frac{S_{\lfloor sn\rfloor}}{\sigma\sqrt{n}}; 0 \leq s \leq \Delta\Big)f(S_n - y);\ \underline{S}_n \geq -a,\ \underline{S}_{[\Delta n,n]} \geq y\Big]$$
$$= C_1 R(a)\int_{x\geq 0} f(x)R_-(x)dx\,\mathbb{E}[F(e_s; 0 \leq s \leq \Delta)], \tag{4.2.10}$$

uniformly in $y \in [0, r_n]$.

(II) Lattice case. *If the distribution of* $(S_1 - S_0)$ *is supported in* $(\alpha + \beta\mathbb{Z})$ *with span* β*, then for any* $d \in \mathbb{R}$*,*

$$\lim_{n\to\infty} n^{3/2}\mathbb{E}\Big[F\Big(\frac{S_{\lfloor sn\rfloor}}{\sigma\sqrt{n}};0\le s\le\Delta\Big)f(S_n-y+d);\ \underline{S}_n\ge -a,\ \underline{S}_{[\Delta n,n]}\ge y-d\Big]$$
$$= C_1R(a)\beta\sum_{j\ge\lceil -\frac{d}{\beta}\rceil} f(\beta j+d)R_-(\beta j+d)\mathbb{E}[F(e_s;0\le s\le\Delta)]. \quad (4.2.11)$$

uniformly in $y\in[0,r_n]\cap\{\alpha n+\beta\mathbb{Z}\}$*.*

Proof. The lemma is a refinement of Lemma 2.3 in [4], which proved the convergence in the non-lattice case when $a=0$ and $F\equiv 1$. We consider the non-lattice case first. We denote the expectation on the left-hand side of (4.2.10) by $\chi(F,f)$. Observe that for any $K\in\mathbb{N}_+$,

$$\chi(F,f)=\chi\big(F,f(x)1_{(0\le x\le K)}\big)+\chi\big(F,f(x)1_{(x>K)}\big).$$

Since $0\le F\le 1$, we have $\chi\big(F,f(x)1_{(x>K)}\big)\le\chi\big(1,f(x)1_{(x>K)}\big)$, which is bounded by

$$\sum_{j\ge K}\mathbb{E}_a\Big[f(S_n-y-a);\ \underline{S}_n\ge 0,\ \underline{S}_{[\Delta n,n]}\ge y+a,\ S_n\in[y+a+j,y+a+j+1]\Big].$$

Recall that $|f(x)|\le\overline{f}(x)$ with $\overline{f}$ non-increasing. We get that

$$\chi\big(1,f(x)1_{(x>K)}\big)\le\sum_{j\ge K}\overline{f}(j)\mathbb{P}_a\Big[\underline{S}_n\ge 0,\ \underline{S}_{[\Delta n,n]}\ge y+a,\ S_n\in[y+a+j,y+a+j+1]\Big].$$

It then follows from (4.2.9) that

$$\chi\big(1,f(x)1_{(x>K)}\big)\le 2c_2(1+a)\Big(\sum_{j\ge K}\overline{f}(j)(2+j)\Big)n^{-3/2}. \quad (4.2.12)$$

Since $\int_0^\infty x\overline{f}(x)dx<\infty$, the sum $\sum_{j\ge K}\overline{f}(j)(2+j)$ decreases to zero as $K\uparrow\infty$. We thus only need to estimate $\chi\big(F,f(x)1_{(0\le x\le K)}\big)$. Note that f is Riemann integrable. It suffices to consider $\chi\Big(F,1_{(0\le x\le K)}\Big)$ with K a positive constant.

Applying the Markov property at time $\lfloor\Delta n\rfloor$ shows that

$$\begin{aligned}\chi\Big(F,1_{(0\le x\le K)}\Big) &= \mathbb{E}_a\Big[F\Big(\frac{S_{\lfloor sn\rfloor}-a}{\sigma\sqrt{n}};0\le s\le\Delta\Big);\ S_n\le y+a+K,\underline{S}_n\ge 0,\underline{S}_{[\Delta n,n]}\ge y+a\Big]\\ &= \mathbb{E}_a\Big[F\Big(\frac{S_{\lfloor sn\rfloor}-a}{\sigma\sqrt{n}};0\le s\le\Delta\Big)\Psi_K(S_{\lfloor\Delta n\rfloor});\ \underline{S}_{\lfloor\Delta n\rfloor}\ge 0\Big], \quad (4.2.13)\end{aligned}$$

where $\Psi_K(x) := \mathbb{P}_x\Big[S_{n-\lfloor\Delta n\rfloor} \leq y+a+K,\ \underline{S}_{n-\lfloor\Delta n\rfloor} \geq y+a\Big]$. By reversing time, we obtain that $\Psi_K(x) = \mathbb{P}\Big[\underline{S}_m^- \geq (-S_m)+(y+a-x) \geq -K\Big]$ with $m := n-\lfloor\Delta n\rfloor$.

We define τ_n as the first time when the random walk $(-S)$ hits the minimal level during $[0,n]$, namely, $\tau_n := \inf\{k\in[0,n] : -S_k = \underline{S}_n^-\}$. Define also $\varkappa(z,\zeta;n) := \mathbb{P}(-S_n\in[z,z+\zeta],\ \underline{S}_n^- \geq 0)$ for any z, $\zeta \geq 0$. Then,

$$\begin{aligned}\Psi_K(x) &= \sum_{k=0}^{m}\mathbb{P}\Big[\tau_m = k;\ \underline{S}_m^- \geq (-S_m)+(y+a-x)\geq -K\Big]\\ &= \sum_{k=0}^{m}\mathbb{E}\Big[1_{\{-S_k=\underline{S}_k^-\geq -K\}}\times\varkappa(x-y-a,\underline{S}_k^-+K;m-k)\Big],\end{aligned}\tag{4.2.14}$$

where the last equality follows from the Markov property at time k.

Let $\psi(x) := xe^{-x^2/2}1_{(x\geq 0)}$. Combining Theorem 1 of [52] with (4.2.6) yields that

$$\varkappa(z,\zeta;n) = \mathbb{P}_0\Big[-S_n\in[z,z+\zeta];\ \underline{S}_n \geq 0\Big] = \frac{C_-\zeta}{\sigma n}\psi\Big(\frac{z}{\sigma\sqrt{n}}\Big)+o(n^{-1}),\tag{4.2.15}$$

uniformly in $z\in\mathbb{R}_+$ and ζ in compact sets of $\mathbb{R}_+$. Note that ψ is bounded on $\mathbb{R}_+$. Therefore, there exists a constant $c_3>0$ such that for any $\zeta\in[0,K]$, $z\geq 0$ and $n\geq 0$,

$$\varkappa(z,\zeta;n) \leq c_3\frac{(1+K)}{n+1}.\tag{4.2.16}$$

Let $k_n := \lfloor\sqrt{n}\rfloor$. We divide the sum on the right-hand side of (4.2.14) into two parts :

$$\Psi_K(x) = \sum_{k=0}^{k_n}+\sum_{k=k_n+1}^{m}\mathbb{P}\big[-S_k=\underline{S}_k^-\geq -K;\ \varkappa(x-y-a,\underline{S}_k^-+K;m-k)\big].\tag{4.2.17}$$

By (4.2.15), under the assumption that $y=o(\sqrt{n})$, the first part becomes that

$$\begin{aligned}&\frac{C_-}{\sigma m}\psi\Big(\frac{x-a}{\sigma\sqrt{m}}\Big)\sum_{k=0}^{k_n}\mathbb{E}\big[\underline{S}_k^-+K;-S_k=\underline{S}_k^-\geq -K\big]+o(n^{-1})\sum_{k=0}^{k_n}\mathbb{P}\big[-S_k=\underline{S}_k^-\geq -K\big]\\ =\ &\frac{C_-}{\sigma m}\psi\Big(\frac{x-a}{\sigma\sqrt{m}}\Big)\int_0^K R_-(u)du+o(n^{-1}),\end{aligned}\tag{4.2.18}$$

where the last equation comes from the fact that $\sum_{k\geq 0}\mathbb{E}\big[\underline{S}_k^-+K;-S_k=\underline{S}_k^-\geq -K\big]=\int_0^K R_-(u)du$. On the other hand, using (4.2.16) for $\varkappa(x-y-a,\underline{S}_k^-+K;m-k)$ and then applying (i) of Fact 4.2.3 imply that for n large enough, the second part of (4.2.17) is bounded by

$$\begin{aligned}&\sum_{k=k_n+1}^{m}c_3\frac{1+K}{m+1-k}\mathbb{P}\big(\underline{S}_k^-\geq -K,\ -S_k\in[-K,0]\big)\\ &\leq c_4\sum_{k=k_n+1}^{m}\frac{(1+K)^3}{(m+1-k)k^{3/2}} = o(n^{-1}).\end{aligned}\tag{4.2.19}$$

By (4.2.18) and (4.2.19), we obtain that as n goes to infinity,

$$\Psi_K(x) = o(n^{-1}) + \frac{C_-}{\sigma(n-\lfloor \Delta n \rfloor)} \psi\Big(\frac{x-a}{\sigma\sqrt{n-\lfloor \Delta n \rfloor}}\Big) \int_0^K R_-(u)du, \tag{4.2.20}$$

uniformly in $x \geq 0$ and $y \in [0, r_n]$. Plugging it into (4.2.13) and then combining with (4.2.6) yield that

$$\begin{aligned}\chi(F, 1_{(0\leq x\leq K)}) \;&=\; o(n^{-3/2}) + \frac{C_-}{\sigma(1-\Delta)n}\int_0^K R_-(u)du \\ &\quad \times \frac{C_+R(a)}{\sqrt{\Delta n}} \mathbb{E}_a\Big[F\Big(\frac{S_{\lfloor sn \rfloor}-a}{\sigma\sqrt{n}}; 0 \leq s \leq \Delta\Big)\psi\Big(\frac{S_{\Delta n}-a}{\sigma\sqrt{(1-\Delta)n}}\Big)\Big|\underline{S}_{\Delta n} \geq 0\Big].\end{aligned}$$

Theorem 1.1 of [53] says that under the conditioned probability $\mathbb{P}_a\Big(\cdot\,\Big|\underline{S}_{\Delta n} \geq 0\Big)$, $(\frac{S_{\lfloor r\Delta n \rfloor}}{\sigma\sqrt{\Delta n}}; 0 \leq r \leq 1)$ converges in law to a Brownian meander, denoted by $(\mathscr{M}_r; 0 \leq r \leq 1)$. Therefore,

$$\chi(F, 1_{(0\leq x\leq K)}) \sim \frac{C_-C_+R(a)}{\sigma n^{3/2}(1-\Delta)\sqrt{\Delta}}\int_0^K R_-(u)du\mathbb{E}\Big[F\Big(\sqrt{\Delta}\mathscr{M}_{s/\Delta}; 0 \leq s \leq \Delta\Big)\psi\Big(\frac{\sqrt{\Delta}\mathscr{M}_1}{\sqrt{1-\Delta}}\Big)\Big].$$

It remains to check that

$$\frac{1}{(1-\Delta)\sqrt{\Delta}}\mathbb{E}\Big[F\Big(\sqrt{\Delta}\mathscr{M}_{s/\Delta}; 0 \leq s \leq \Delta\Big)\psi\Big(\frac{\sqrt{\Delta}\mathscr{M}_1}{\sqrt{1-\Delta}}\Big)\Big] = \sqrt{\frac{\pi}{2}}\mathbb{E}\Big[F(e_s; 0 \leq s \leq \Delta)\Big]. \tag{4.2.21}$$

Let $(R_s; 0 \leq s \leq 1)$ be a standard three-dimensional Bessel process. Then, as is shown in [88],

$$\begin{aligned}&\frac{1}{(1-\Delta)\sqrt{\Delta}}\mathbb{E}\Big[F\Big(\sqrt{\Delta}\mathscr{M}_{s/\Delta}; 0 \leq s \leq \Delta\Big)\psi\Big(\frac{\sqrt{\Delta}\mathscr{M}_1}{\sqrt{1-\Delta}}\Big)\Big] \\ =\;& \sqrt{\frac{\pi}{2}}\frac{1}{(1-\Delta)\sqrt{\Delta}}\mathbb{E}\Big[\frac{1}{R_1}F\Big(\sqrt{\Delta}R_{s/\Delta}; 0 \leq s \leq \Delta\Big)\psi\Big(\frac{\sqrt{\Delta}R_1}{\sqrt{1-\Delta}}\Big)\Big], \\ =\;& \sqrt{\frac{\pi}{2}}\mathbb{E}\Big[\frac{1}{(1-\Delta)^{3/2}}e^{-\frac{R_\Delta^2}{2(1-\Delta)}}F\Big(R_s; 0 \leq s \leq \Delta\Big)\Big],\end{aligned}$$

where the last equation follows from the scaling property of Bessel process. Let $(r_s; 0 \leq s \leq 1)$ be a standard three-dimensional Bessel bridge. Note that for any $\Delta < 1$, $(r_s; 0 \leq s \leq \Delta)$ is equivalent to $(R_s; 0 \leq s \leq \Delta)$, with density $(1-\Delta)^{-3/2}\exp(-\frac{R_\Delta^2}{2(1-\Delta)})$ (see p. 468 (3.11) of [126]). Thus,

$$\frac{1}{(1-\Delta)\sqrt{\Delta}}\mathbb{E}\Big[F\Big(\sqrt{\Delta}\mathscr{M}_{s/\Delta}; 0 \leq s \leq \Delta\Big)\psi\Big(\frac{\sqrt{\Delta}\mathscr{M}_1}{\sqrt{1-\Delta}}\Big)\Big] = \sqrt{\frac{\pi}{2}}\mathbb{E}\Big[F(r_s; 0 \leq s \leq \Delta)\Big].$$

Since a normalized Brownian excursion is exactly a standard three-dimensional Bessel bridge, this yields (4.2.21). Therefore, (4.2.10) is proved by taking $C_1 = \sqrt{\frac{\pi}{2}}\frac{C_-C_+}{\sigma}$.

The proof of the lemma in the lattice case is along the same lines, except that we use Theorem 2 (instead of Theorem 1) of [52]. $\square$

4.3 Conditioning on the event $\{I_n \le \frac{3}{2}\log n - z\}$

On the event $\{I_n \le \frac{3}{2}\log n - z\}$, we analyze the sample path leading to a particle located at the leftmost position at the nth generation. For $z \ge 0$ and $n \ge 1$, let $a_n(z) := \frac{3}{2}\log n - z$ if the distribution of $\mathscr{L}$ is non-lattice and let $a_n(z) := \alpha n + \beta\lfloor \frac{\frac{3}{2}\log n - \alpha n}{\beta}\rfloor - z$ if the distribution of $\mathscr{L}$ is supported by $\alpha + \beta\mathbb{Z}$. This section is devoted to the proof of the following proposition.

Proposition 4.3.1. *For any $\Delta \in (0,1]$ and any continuous functional $F : D([0,\Delta],\, \mathbb{R}) \to [0,1]$,*

$$\lim_{z\to\infty}\limsup_{n\to\infty}\left|\mathbb{E}\Big[F\Big(\frac{I_n(\lfloor sn\rfloor)}{\sigma\sqrt{n}};0\le s\le\Delta\Big)\Big|I_n\le a_n(z)\Big]-\mathbb{E}\Big[F(e_s;0\le s\le\Delta)\Big]\right|=0. \tag{4.3.1}$$

We begin with some preliminary results.

For any $0<\Delta<1$ and L, $K\ge 0$, we denote by $J^{\Delta}_{z,K,L}(n)$ the following collection of particles :

$$\Big\{u\in\mathbb{T}:|u|=n,\ V(u)\le a_n(z),\ \min_{0\le k\le n}V(u_k)\ge -z+K,\ \min_{\Delta n\le k\le n}V(u_k)\ge a_n(z+L)\Big\}. \tag{4.3.2}$$

Lemma 4.3.2. *For any $\varepsilon>0$, there exists $L_\varepsilon>0$ such that for any $L\ge L_\varepsilon$, $n\ge 1$ and $z\ge K\ge 0$,*

$$\mathbb{P}\Big(m^{(n)}\notin J^{\Delta}_{z,K,L}(n),\ I_n\le a_n(z)\Big)\le\Big(e^K+\varepsilon(1+z-K)\Big)e^{-z}. \tag{4.3.3}$$

Proof. It suffices to show that for any $\varepsilon\in(0,1)$, there exists $L_\varepsilon\ge 1$ such that for any $L\ge L_\varepsilon$, $n\ge 1$ and $z\ge K\ge 0$,

$$\mathbb{P}\Big(\exists|u|=n:V(u)\le a_n(z),\ u\notin J^{\Delta}_{z,K,L}(n)\Big)\le\Big(e^K+\varepsilon(1+z-K)\Big)e^{-z}. \tag{4.3.4}$$

We observe that

$$\begin{aligned}\mathbb{P}\Big(\exists|u|=n:V(u)\le a_n(z),\ u\notin J^{\Delta}_{z,K,L}(n)\Big)&\le\mathbb{P}\Big(\exists u\in\mathbb{T}:V(u)\le -z+K\Big)\\&+\mathbb{P}\Big(\exists|u|=n:V(u)\le a_n(z),\ \min_{0\le k\le n}V(u_k)\ge -z+K,\ \min_{\Delta n\le k\le n}V(u_k)\le a_n(z+L)\Big).\end{aligned} \tag{4.3.5}$$

On the one hand, by (4.2.4) (i.e., the many-to-one lemma),

$$\begin{aligned}\mathbb{P}\Big(\exists u\in\mathbb{T}:V(u)\le -z+k\Big)&\le\sum_{n\ge0}\mathbb{E}\Big[\sum_{|u|=n}1_{\{V(u)\le -z+K<\min_{k<n}V(u_k)\}}\Big]\\&=\sum_{n\ge0}\mathbb{E}[e^{S_n};\ S_n\le -z+K<\underline{S}_{n-1}]\le e^{-z+K}.\end{aligned} \tag{4.3.6}$$

On the other hand, denoting $A_n(z) := [a_n(z)-1, a_n(z)]$ for any $z \ge 0$,

$$\begin{aligned}
&\mathbb{P}\Big(\exists |u| = n : V(u) \le a_n(z),\ \min_{0\le k\le n} V(u_k) \ge -z+K,\ \min_{\Delta n\le k\le n} V(u_k) \le a_n(z+L)\Big)\\
=\ &\mathbb{P}_{z-K}\Big(\exists |u| = n : V(u) \le a_n(K),\ \min_{0\le k\le n} V(u_k) \ge 0,\ \min_{\Delta n\le k\le n} V(u_k) \le a_n(K+L)\Big)\\
\le\ &\sum_{\ell\ge L+K}\sum_{j=K}^{j=K+\ell} \mathbb{P}_{z-K}\Big(\exists |u| = n : V(u) \in A_n(j),\ \min_{0\le k\le n} V(u_k) \ge 0,\ \min_{\Delta n\le k\le n} V(u_k) \in A_n(\ell)\Big).
\end{aligned}$$

According to Lemma 3.3 in [4], there exist constants $1 > c_5 > 0$ and $c_6 > 0$ such that for any $n \ge 1$, $L \ge 0$ and $x, z \ge 0$,

$$\mathbb{P}_x\Big(\exists u \in \mathbb{T} : |u| = n,\ V(u) \in A_n(z),\ \min_{0\le k\le n} V(u_k) \ge 0,\ \min_{\Delta n\le k\le n} V(u_k) \in A_n(z+L)\Big) \le c_6(1+x)e^{-c_5 L}e^{-x-z}. \quad (4.3.7)$$

Hence, combining (4.3.6) with (4.3.5) yields that

$$\begin{aligned}
&\mathbb{P}\Big(\exists |u| = n : V(u) \le a_n(z),\ u \notin J^{\Delta}_{z,K,L}(n)\Big)\\
\le\ & e^{-z+K} + \sum_{\ell\ge L}\sum_{0\le j\le \ell} c_6(1+z-K)e^{-c_5(\ell-j)}e^{-z-j}\\
\le\ & \Big(e^K + c_7\sum_{\ell\ge L} e^{-c_5\ell}(1+z-K)\Big)e^{-z},
\end{aligned}$$

where the last inequality comes from the fact that $\sum_{j\ge 0} e^{-(1-c_5)j} < \infty$. We take $L_\varepsilon = -c_8\log\varepsilon$ so that $c_7\sum_{\ell\ge L} e^{-c_5\ell} \le \varepsilon$ for all $L \ge L_\varepsilon$. Therefore, for any $L \ge L_\varepsilon$, $n \ge 1$ and $z \ge K \ge 0$,

$$\mathbb{P}\Big(\exists |u| = n : V(u) \le a_n(z),\ u \notin J^{\Delta}_{z,K,L}(n)\Big) \le \Big(e^K + \varepsilon(1+z-K)\Big)e^{-z}, \quad (4.3.8)$$

which completes the proof. □

Recall that w_k is the k-th particle in the spine of $\mathbb{T}$. For $b \in \mathbb{Z}_+$, we define

$$\mathscr{E}_n = \mathscr{E}_n(z,b) := \{\forall k \le n-b,\ \min_{u\ge w_k, |u|=n} V(u) > a_n(z)\}. \quad (4.3.9)$$

We note that on the event $\mathscr{E}_n \cap \{I_n \le a_n(z)\}$, any particle located at the leftmost position must be separated from the spine after time $n-b$.

Lemma 4.3.3. *For any $\eta > 0$ and $L > 0$, there exist $K(\eta) > 0$, $B(L,\eta) \ge 1$ and $N(\eta) \ge 1$ such that for any $b \ge B(L,\eta)$, $n \ge N(\eta)$ and $z \ge K \ge K(\eta)$,*

$$\mathbb{Q}\Big(\mathscr{E}_n^c,\ w_n \in J^{\Delta}_{z,K,L}(n)\Big) \le \eta(1+L)^2(1+z-K)n^{-3/2}. \quad (4.3.10)$$

We feel free to omit the proof of Lemma 4.3.3 since it is just a slightly stronger version of Lemma 3.8 in [4]. It follows from the same arguments.

Let us turn to the proof of Proposition 4.3.1. We break it up into 3 steps.

Step (I) (The conditioned convergence of $(\frac{I_n(\lfloor sn\rfloor)}{\sigma\sqrt{n}};0\le s\le\Delta)$ *for* $\Delta<1$ *in the non-lattice case)*

Assume that the distribution of $\mathscr{L}$ is non-lattice in this step. Recall that $a_n(z)=\frac{3}{2}\log n-z$. The tail distribution of I_n has been given in Propositions 1.3 and 4.1 of [4], recalled as follows.

Fact 4.3.4 ([4]). *There exists a constant* $C>0$ *such that*

$$\lim_{z\to\infty}\limsup_{n\to\infty}\Big|\frac{e^z}{z}\mathbb{P}(I_n\le a_n(z))-C\Big|=0. \tag{4.3.11}$$

Furthermore, for any $\varepsilon>0$*, there exist* $N_\varepsilon\ge1$ *and* $\Lambda_\varepsilon>0$ *such that for any* $n\ge N_\varepsilon$ *and* $\Lambda_\varepsilon\le z\le\frac{3}{2}\log n-\Lambda_\varepsilon$*,*

$$\Big|\frac{e^z}{z}\mathbb{P}(I_n\le a_n(z))-C\Big|\le\varepsilon.$$

For any continuous functional $F:D([0,\Delta],\,\mathbb{R})\to[0,1]$, it is convenient to write that

$$\Sigma_n(F,z):=\mathbb{E}\Big[F\Big(\frac{I_n(\lfloor sn\rfloor)}{\sigma\sqrt{n}};0\le s\le\Delta\Big)1_{\{I_n\le a_n(z)\}}\Big]. \tag{4.3.12}$$

In particular, if $F\equiv1$, $\Sigma_n(1,z)=\mathbb{P}(I_n\le a_n(z))$. Thus,

$$\frac{\Sigma_n(F,z)}{\Sigma_n(1,z)}=\mathbb{E}\Big[F\Big(\frac{I_n(\lfloor sn\rfloor)}{\sigma\sqrt{n}};0\le s\le\Delta\Big)\Big|I_n\le a_n(z)\Big]. \tag{4.3.13}$$

Let us prove the following convergence for $0<\Delta<1$,

$$\lim_{z\to\infty}\limsup_{n\to\infty}\Big|\frac{\Sigma_n(F,z)}{\Sigma_n(1,z)}-\mathbb{E}[F(e_s,\ 0\le s\le\Delta)]\Big|=0. \tag{4.3.14}$$

Proof. For any $n\ge1$, $L\ge0$ and $z\ge K\ge0$, let

$$\Pi_n(F)=\Pi_n(F,z,K,L):=\mathbb{E}\Big[F\Big(\frac{I_n(sn)}{\sigma\sqrt{n}};0\le s\le\Delta\Big)1_{\{m^{(n)}\in J^\Delta_{z,K,L}(n)\}}\Big]. \tag{4.3.15}$$

By Lemma 4.3.2, we obtain that for $L\ge L_\varepsilon$, $n\ge1$ and $z\ge K\ge0$,

$$\Big|\Sigma_n(F,z)-\Pi_n(F)\Big|\le\Big(e^K+\varepsilon(1+z-K)\Big)e^{-z}. \tag{4.3.16}$$

Note that $m^{(n)}$ is chosen uniformly among the particles located at the leftmost position. Thus,

$$\begin{aligned}\Pi_n(F)&=\mathbb{E}\Big[\sum_{|u|=n}1_{(u=m^{(n)},\,u\in J^\Delta_{z,K,L}(n))}F\Big(\frac{V(u_{\lfloor sn\rfloor})}{\sigma\sqrt{n}};0\le s\le\Delta\Big)\Big]\\&=\mathbb{E}\Big[\frac{1}{\sum_{|u|=n}1_{(V(u)=I_n)}}\sum_{|u|=n}1_{(V(u)=I_n,\,u\in J^\Delta_{z,K,L}(n))}F\Big(\frac{V(u_{\lfloor sn\rfloor})}{\sigma\sqrt{n}};0\le s\le\Delta\Big)\Big].\end{aligned}$$

Applying the change of measures given in (4.2.2), it follows from Proposition 4.2.1 that

$$\Pi_n(F) = \mathbf{E}_{\mathbb{Q}}\Big[\frac{e^{V(w_n)}}{\sum_{|u|=n} 1_{(V(u)=I_n)}} 1_{(V(w_n)=I_n,\, w_n\in J^{\Delta}_{z,K,L}(n))} F\Big(\frac{V(w_{\lfloor sn\rfloor})}{\sigma\sqrt{n}};0\leq s\leq \Delta\Big)\Big]. \qquad (4.3.17)$$

In order to estimate Π_n, we restrict ourselves to the event $\mathscr{E}_n$. Define

$$\Lambda_n(F) := \mathbf{E}_{\mathbb{Q}}\Big[\frac{e^{V(w_n)}}{\sum_{|u|=n} 1_{(V(u)=I_n)}} 1_{(V(w_n)=I_n,\, w_n\in J^{\Delta}_{z,K,L}(n))} F\Big(\frac{V(w_{\lfloor sn\rfloor})}{\sigma\sqrt{n}};0\leq s\leq \Delta\Big);\ \mathscr{E}_n\Big].$$

In view of Lemma 4.3.3, for any $b\geq B(L,\eta)$, $n\geq N(\eta)$ and $z\geq K\geq K(\eta)$,

$$\begin{aligned}\Big|\Pi_n(F)-\Lambda_n(F)\Big| &\leq \mathbf{E}_{\mathbb{Q}}\Big[e^{V(w_n)};\ w_n\in J^{\Delta}_{z,K,L}(n),\, \mathscr{E}_n^c\Big] \qquad (4.3.18)\\ &\leq e^{-z}n^{-3/2}\mathbb{Q}\Big(\mathscr{E}_n^c,\, w_n\in J^{\Delta}_{z,K,L}(n)\Big)\\ &\leq \eta(1+L)^2(1+z-K)e^{-z}.\end{aligned}$$

On the event $\mathscr{E}_n\cap\{I_n\leq a_n(z)\}$, $\Lambda_n(F)$ equals

$$\mathbf{E}_{\mathbb{Q}}\Big[\frac{e^{V(w_n)}}{\sum_{u>w_{n-b},|u|=n} 1_{(V(u)=I_n)}} 1_{(V(w_n)=I_n,\, w_n\in J^{\Delta}_{z,K,L}(n))} F\Big(\frac{V(w_{\lfloor sn\rfloor})}{\sigma\sqrt{n}};0\leq s\leq \Delta\Big);\ \mathscr{E}_n\Big].$$

Let, for $x\geq 0$, $L>0$, and $b\geq 1$,

$$\begin{aligned} f_{L,b}(x) &:= \mathbb{E}_{\mathbb{Q}_x}\Big[\frac{e^{V(w_b)-L}1_{\{V(w_b)=I_b\}}}{\sum_{|u|=b}1_{\{V(u)=I_b\}}}, \min_{0\leq k\leq b}V(w_k)\geq 0, V(w_b)\leq L\Big]\\ &\leq \mathbb{Q}_x\Big(\min_{0\leq k\leq b}V(w_k)\geq 0, V(w_b)\leq L\Big). \qquad (4.3.19)\end{aligned}$$

We choose n large enough so that $\Delta n\leq n-b$. Thus, applying the Markov property at time $n-b$ yields that

$$\begin{aligned}\Lambda_n(F) = n^{3/2}e^{-z}\mathbf{E}_{\mathbb{Q}}\Big[F\Big(\frac{V(w_{\lfloor sn\rfloor})}{\sigma\sqrt{n}};0\leq s\leq\Delta\Big)f_{L,b}(V(w_{n-b})-a_n(z+L));\\ \min_{0\leq k\leq n-b}V(w_k)\geq -z+K,\ \min_{\Delta n\leq k\leq n-b}V(w_k)\geq a_n(z+L),\ \mathscr{E}_n\Big]. \qquad (4.3.20)\end{aligned}$$

Let us introduce the following quantity by removing the restriction to $\mathscr{E}_n$:

$$\begin{aligned}\Lambda^I_n(F) := n^{3/2}e^{-z}\mathbf{E}_{\mathbb{Q}}\Big[F\Big(\frac{V(w_{\lfloor sn\rfloor})}{\sigma\sqrt{n}};0\leq s\leq\Delta\Big)f_{L,b}(V(w_{n-b})-a_n(z+L));\\ \min_{0\leq k\leq n-b}V(w_k)\geq -z+K,\ \min_{\Delta n\leq k\leq n-b}V(w_k)\geq a_n(z+L)\Big]. \qquad (4.3.21)\end{aligned}$$

We immediately observe that

$$\Big|\Lambda_n(F)-\Lambda_n^I(F)\Big| \le n^{3/2}e^{-z}\mathbb{Q}\Big(f_{L,b}(V(w_{n-b})-a_n(z+L)),$$
$$\min_{0\le k\le n-b} V(w_k)\ge -z+K,\ \min_{\Delta n\le k\le n-b} V(w_k)\ge a_n(z+L);\ (\mathscr{E}_n)^c\Big). \quad (4.3.22)$$

By (4.3.19), we check that $\Big|\Lambda_n(F)-\Lambda_n^I(F)\Big| \le n^{3/2}e^{-z}\mathbb{Q}(w_n\in J^{\Delta}_{z,K,L}(n),\ (\mathscr{E}_n)^c)$. Applying Lemma 4.3.3 again implies that

$$\Big|\Lambda_n(F)-\Lambda_n^I(F)\Big| \le \eta(1+L)^2(1+z-K)e^{-z}. \quad (4.3.23)$$

Combining with (4.3.18), we obtain that for any $b\ge B(L,\eta)$, $z\ge K\ge K(\eta)$ and n large enough,

$$\Big|\Pi_n(F)-\Lambda_n^I(F)\Big| \le 2\eta(1+L)^2(1+z-K)e^{-z}. \quad (4.3.24)$$

Note that $(V(w_k);\ k\ge 1)$ is a centered random walk under $\mathbb{Q}$ and that it is proved in [4] that $f_{L,b}$ satisfies the conditions of Lemma 4.2.4. By (I) of Lemma 4.2.4, we get that

$$\lim_{n\to\infty}\Lambda_n^I(F)=\alpha^I_{L,b}R(z-K)e^{-z}\mathbb{E}[F(e_s,\ 0\le s\le \delta)], \quad (4.3.25)$$

where $\alpha^I_{L,b}:=C_1\int_{x\ge 0}f_{L,b}(x)R_-(x)dx\in[0,\infty)$. Thus, by (4.3.24), one sees that for any $b\ge B(L,\eta)$ and $z\ge K\ge K(\eta)$,

$$\limsup_{n\to\infty}\Big|\Pi_n(F)-\alpha^I_{L,b}R(z-K)e^{-z}\mathbb{E}[F(e_s,\ 0\le s\le\Delta)]\Big| \le 2\eta(1+L)^2(1+z-K)e^{-z}. \quad (4.3.26)$$

Going back to (4.3.16), we deduce that for any $L\ge L_\varepsilon$, $b\ge B(L,\eta)$ and $z\ge K\ge K(\eta)$,

$$\begin{aligned}&\limsup_{n\to\infty}\Big|\Sigma_n(F,z)-\alpha^I_{L,b}R(z-K)e^{-z}\mathbb{E}[F(e_s,\ 0\le s\le\Delta)]\Big|\\ \le\ & 2\eta(1+L)^2(1+z-K)e^{-z}+\Big(e^K+\varepsilon(1+z-K)\Big)e^{-z}.\end{aligned}$$

Recall that $\lim_{z\to\infty}\frac{R(z)}{z}=c_0$. We multiply each term by $\frac{e^z}{z}$, and then let z go to infinity to conclude that

$$\limsup_{z\to\infty}\limsup_{n\to\infty}\Big|\frac{e^z}{z}\Sigma_n(F,z)-\alpha^I_{L,b}c_0\mathbb{E}[F(e_s,\ 0\le s\le\Delta)]\Big|\le 2\eta(1+L)^2+\varepsilon. \quad (4.3.27)$$

In particular, taking $F\equiv 1$ gives that

$$\limsup_{z\to\infty}\limsup_{n\to\infty}\Big|\frac{e^z}{z}\mathbb{P}(I_n\le a_n(z))-\alpha^I_{L,b}c_0\Big|\le 2\eta(1+L)^2+\varepsilon. \quad (4.3.28)$$

It follows from Fact 4.3.4 that $|C-\alpha^I_{L,b}c_0| \le 2\eta(1+L)^2+\varepsilon$. We thus choose $0<\varepsilon<C/10$ and $0<\eta\le\frac{\varepsilon}{2(1+L_\varepsilon)^2}$ so that $2C>\alpha^I_{L_\varepsilon,b}c_0>C/2>0$.

Therefore, for any $\varepsilon\in(0,C/10)$, $0<\eta\le\frac{\varepsilon}{2(1+L_\varepsilon)^2}$, $L=L_\varepsilon$ and $b\ge B(L_\varepsilon,\eta)$,

$$\limsup_{z\to\infty}\limsup_{n\to\infty}\left|\frac{\Sigma_n(F,z)}{\Sigma_n(1,z)}-\mathbb{E}[F(e_s,0\le s\le\Delta)]\right|\le\frac{4\varepsilon}{C/2-2\varepsilon}, \tag{4.3.29}$$

which completes the proof of (4.3.14) in the non-lattice case. □

Step (II) (The conditioned convergence of $(\frac{I_n(sn)}{\sigma\sqrt{n}};0\le s\le\Delta)$ *for* $\Delta<1$ *in the lattice case)* Assume that the law of $\mathscr{L}$ is supported by $\alpha+\beta\mathbb{Z}$ with span β. Recall that $a_n(0)=\alpha n+\beta\lfloor\frac{\frac{3}{2}\log n-\alpha n}{\beta}\rfloor$ and that $a_n(z)=a_n(0)-z$. We use the same notation of Step (I). Let us prove

$$\lim_{\beta\mathbb{Z}\ni z\to\infty}\limsup_{n\to\infty}\left|\frac{\Sigma_n(F,z)}{\Sigma_n(1,z)}-\mathbb{E}[F(e_s,\ 0\le s\le\Delta)]\right|=0. \tag{4.3.30}$$

Proof. Suppose that $z\in\beta\mathbb{Z}$. Whereas the arguments of Step (I), we obtain that for any $L\ge L_\varepsilon$, $b\ge B(L,\eta)$, $z\ge K\ge K(\eta)$ and n sufficiently large,

$$\left|\Sigma_n(F,z)-\Lambda^{II}_n(F)\right|\le 2\eta(1+L)^2(1+z-K)e^{-z}+\Big(e^K+\varepsilon(1+z-K)\Big)e^{-z}, \tag{4.3.31}$$

where

$$\Lambda^{II}_n(F)=\Lambda^{II}(F,z,K,L,b):=e^{a_n(0)}e^{-z}\mathbf{E}_{\mathbb{Q}}\Big[F\Big(\frac{V(w_{\lfloor sn\rfloor})}{\sigma\sqrt{n}};0\le s\le\Delta\Big)\times$$
$$f_{L,b}\big(V(w_{n-b}-a_n(z+L))\big);\ \min_{0\le k\le n-b}V(w_k)\ge -z+K,\ \min_{\Delta n\le k\le n-b}V(w_k)\ge a_n(z+L)\Big].$$

Under $\mathbb{Q}$, the distribution of $V(w_1)-V(w_0)$ is also supported by $\alpha+\beta\mathbb{Z}$. Let $d=d(L,b):=\beta\lceil\frac{\alpha b-L}{\beta}\rceil-\alpha b+L$ and $\lambda_n:=n^{3/2}e^{-a_n(0)}$. Recall that $f_{L,b}$ is well defined in (4.3.19), it follows from (II) of Lemma 4.2.4 that

$$\lim_{n\to\infty}\lambda_n\Lambda^{II}_n(F)=\alpha^{II}_{L,b}R(z-K)e^{-z}\mathbb{E}[F(e_s,\ 0\le s\le\Delta)]. \tag{4.3.32}$$

where $\alpha^{II}_{L,b}:=C_1\beta\sum_{j\ge0}f_{L,b}(\beta j+d)R_-(\beta j+d)\in[0,\infty)$. Observe that $1\le\lambda_n\le e^\beta$. Combining with (4.3.31), we conclude that

$$\limsup_{\beta\mathbb{Z}\ni z\to\infty}\limsup_{n\to\infty}\left|\frac{e^z}{z}\lambda_n\Sigma_n(F,z)-\alpha^{II}_{L,b}c_0\mathbb{E}[F(e_s,\ 0\le s\le\Delta)]\right|\le e^\beta(2\eta(1+L)^2+\varepsilon). \tag{4.3.33}$$

We admit for the moment that there exist $0<c_9<c_{10}<\infty$ such that $\alpha^{II}_{L,b}\in[c_9,c_{10}]$ for all L, b large enough. Then take $\varepsilon<\frac{c_9c_0}{4e^\beta}$, $L=L_\varepsilon$, $\eta=\frac{\varepsilon}{2(1+L_\varepsilon)^2}$ and $b\ge B(L_\varepsilon,\eta)$ so that $e^\beta(2\eta(1+L)^2+$

$\varepsilon) < c_9c_0/2 \le \alpha^{II}_{L_\varepsilon,b}c_0/2 \le 2c_{10}c_0$. Note that $\frac{\Sigma_n(F,z)}{\Sigma_n(1,z)} = \frac{\frac{e^z}{z}\lambda_n\Sigma_n(F,z)}{\frac{e^z}{z}\lambda_n\Sigma_n(1,z)}$. We thus deduce from (4.3.33) that

$$\limsup_{\beta\mathbb{Z}\ni z\to\infty}\limsup_{n\to\infty}\Big|\frac{\Sigma_n(F,z)}{\Sigma_n(1,z)} - \mathbb{E}[F(e_s,\ 0\le s\le\Delta)]\Big| \le \frac{4\varepsilon}{c_9c_0/e^\beta - 2\varepsilon}, \tag{4.3.34}$$

which tends to zero as $\varepsilon\downarrow 0$.

It remains to prove that $\alpha^{II}_{L,b}\in[c_9,c_{10}]$ for all L, b large enough. Instead of investigating the entire system, we consider the branching random walk killed at 0. Define

$$I_n^{kill} := \inf\{V(u): |u|=n, V(u_k)\ge 0,\ \forall 0\le k\le n\}, \tag{4.3.35}$$

and we get the following fact from Corollary 3.4 and Lemma 3.6 of [4].

Fact 4.3.5 ([4]). *There exists a constant $c_{11}>0$ such that for any $n\ge 1$ and x, $z\ge 0$,*

$$\mathbb{P}_x(I_n^{kill}\le a_n(z)) \le c_{11}(1+x)e^{-x-z}. \tag{4.3.36}$$

Moreover, there exists $c_{12}>0$ such that for any $n\ge 1$ and $z\in[0,a_n(1)]$,

$$\mathbb{P}(I_n^{kill}\le a_n(z)) \ge c_{12}e^{-z}. \tag{4.3.37}$$

Even though Fact 4.3.5 is proved in [4] under the assumption that the distribution of $\mathscr{L}$ is non-lattice, the lattice case is actually recovered from that proof.

Analogically, let $m^{kill,(n)}$ be the particle chosen uniformly in the set $\{u: |u|=n,\ V(u) = I_n^{kill},\ \min_{0\le k\le n}V(u_k)\ge 0\}$. Moreover, let $\Sigma_n^{kill}(1,z) := \mathbb{P}\Big[I_n^{kill}\le a_n(z)\Big]$ and $\Pi_n^{kill}(1,z,z,L) := \mathbb{P}\Big[I_n^{kill}\le a_n(z),\ m^{kill,(n)}\in J^\Delta_{z,z,L}(n)\Big]$. By (4.3.7) again, we check that for all $L\ge L_\varepsilon$,

$$\begin{aligned} &\Big|\Sigma_n^{kill}(1,z) - \Pi_n^{kill}(1,z,z,L)\Big| \\ \le\ & \mathbb{P}\Big[\exists|u|=n: V(u)\le a_n(z);\ \min_{0\le k\le n}V(u_k)\ge 0;\ \min_{\Delta n\le k\le n}V(u_k)\le a_n(z+L)\Big] \\ \le\ & \varepsilon e^{-z}. \end{aligned} \tag{4.3.38}$$

Recounting the arguments of Step (1), one sees that for any $L\ge L_\varepsilon$, $b\ge B(L,\eta)$, $z\ge K(\eta)$ and n sufficiently large,

$$\Big|\Pi_n^{kill}(1,z,z,L) - \Lambda_n^{kill}\Big| \le 2\eta(1+L)^2e^{-z}, \tag{4.3.39}$$

where

$$\Lambda_n^{kill} := \mathbf{E}_{\mathbb{Q}}\Big[f^{kill}(V(w_{n-b}));\ \min_{0\le k\le n-b}V(w_k)\ge 0,\ \min_{\Delta n\le k\le n-b}V(w_k)\ge a_n(z+L)\Big], \tag{4.3.40}$$

with $f^{kill}(x) := \mathbb{E}_{\mathbb{Q}_x}\Big[\frac{e^{V(w_b)}1_{\{V(w_b)=I_b^{kill}\}}}{\sum_{|u|=b}1_{\{V(u)=I_b^{kill},\,\min_{0\le j\le b}V(u_j)\ge 0\}}};\ \min_{0\le k\le b}V(w_k)\ge a_n(z+L), V(w_b)\le a_n(z)\Big]$.
For $\varepsilon>0$ and n sufficiently large, it has been proved in [4] that

$$\left|e^z\Lambda_n^{II}(1,z,z,L,b)-\Lambda_n^{kill}\right|\le\varepsilon. \tag{4.3.41}$$

Recalling the convergence (4.3.32) with $K=z$ and $F\equiv 1$, we deduce from (4.3.38), (4.3.39) and (4.3.41) that for any $L\ge L_\varepsilon$, $b\ge B(L,\eta)$ and $z\ge K(\eta)$,

$$\limsup_{n\to\infty}\left|\lambda_n\Sigma_n^{kill}(1,z)-\alpha_{L,b}^{II}e^{-z}\right|\le e^{\beta}\Big(2\eta(1+L)^2+2\varepsilon\Big)e^{-z}, \tag{4.3.42}$$

since $R(0)=1$ and $1\le\lambda_n\le e^{\beta}$. Fact 4.3.5 implies that $c_{12}\le e^z\lambda_n\mathbb{P}(I_n^{kill}\le a_n(z))\le c_{11}e^{\beta}$. Hence, we obtain that

$$c_{12}-e^{\beta}\Big(2\eta(1+L)^2+2\varepsilon\Big)\le\alpha_{L,b}^{II}\le e^{\beta}c_{11}+e^{\beta}\Big(2\eta(1+L)^2+2\varepsilon\Big). \tag{4.3.43}$$

Let $c_{10}:=c_{11}e^{\beta}+c_{12}$ and $c_9:=3c_{12}/4>0$. For any $\varepsilon<e^{-\beta}c_{12}/12$, we take $L=L_\varepsilon$ and $\eta\le\varepsilon/2(1+L_\varepsilon)^2$. Then $c_{10}>\alpha_{L,b}^{II}\ge c_9>0$ for $b\ge B(L_\varepsilon,\eta)$. This completes the second step. □

Step (III)(The tightness) Actually, it suffices to prove the following proposition.

Proposition 4.3.6. *For any $\eta>0$,*

$$\lim_{\delta\to 0}\limsup_{z\to\infty}\limsup_{n\to\infty}\mathbb{P}\Big(\sup_{0\le k\le\delta n}|I_n(n-k)-I_n|\ge\eta\sigma\sqrt{n}\Big|I_n\le a_n(z)\Big)=0. \tag{4.3.44}$$

The first two steps allow us to obtain the following fact whether the distribution is lattice or non-lattice.

Fact 4.3.7. *There exist constants $c_{13},c_{14}\in(0,\infty)$ such that*

$$c_{13}\le\liminf_{z\to\infty}\liminf_{n\to\infty}\frac{e^z}{z}\mathbb{P}(I_n\le a_n(z))\le\limsup_{z\to\infty}\limsup_{n\to\infty}\frac{e^z}{z}\mathbb{P}(I_n\le a_n(z))\le c_{14}. \tag{4.3.45}$$

Proof of Proposition 4.3.6. First, we observe that for any $M\ge 1$ and $\delta\in(0,1/2)$,

$$\begin{aligned}&\mathbb{P}\Big(\sup_{0\le k\le\delta n}|I_n(n-k)-I_n|\ge\delta\sigma\sqrt{n},\ I_n\le a_n(z)\Big)\\ \le\ &\mathbb{P}\Big(m_n^{(n)}\notin J_{z,0,L}^{1/2}(n), I_n\le a_n(z)\Big)+\mathbb{P}\Big(I_n(n-\lfloor\delta n\rfloor)\ge M\sigma\sqrt{\delta n}, I_n\le a_n(z)\Big)+\chi(\delta,z,n).\end{aligned}$$

where $\chi(\delta,z,n):=\mathbb{P}\Big(m_n^{(n)}\in J_{z,0,L}^{1/2}(n), I_n(n-\lfloor\delta n\rfloor)\le M\sigma\sqrt{\delta n}, \sup_{0\le k\le\delta n}|I_n(n-k)-I_n|\ge\eta\sigma\sqrt{n}\Big)$.

It follows from Lemma 4.3.2 that for any $\varepsilon>0$, if $L\geq L_\varepsilon$, $n\geq 1$ and $z\geq 0$,

$$\mathbb{P}\Big(m_n^{(n)}\notin J_{z,0,L}^{1/2}(n), I_n\leq a_n(z)\Big)\leq (1+\varepsilon(1+z))e^{-z}. \tag{4.3.46}$$

Then dividing each term of (4.3.46) by $\mathbb{P}(I_n\leq a_n(z))$ yields that

$$\begin{aligned}&\mathbb{P}\Big(\sup_{0\leq k\leq \delta n}|I_n(n-k)-I_n|\geq \eta\sigma\sqrt{n}\Big|I_n\leq a_n(z)\Big) \qquad (4.3.47)\\ \leq\ & \frac{(1+\varepsilon(1+z))e^{-z}}{\mathbb{P}(I_n\leq a_n(z))}+\mathbb{P}\Big(I_n(n-\lfloor\delta n\rfloor)\geq M\sigma\sqrt{\delta n}\Big|I_n\leq a_n(z)\Big)+\frac{\chi(\delta,z,n)}{\mathbb{P}(I_n\leq a_n(z))}.\end{aligned}$$

On the one hand, by Fact 4.3.7,

$$\limsup_{z\to\infty}\limsup_{n\to\infty}\frac{(1+\varepsilon(1+z))e^{-z}}{\mathbb{P}(I_n\leq a_n(z))}\leq\frac{\varepsilon}{c_{13}}. \tag{4.3.48}$$

On the other hand, Steps (I) and (II) tell us that for any $1>\delta>0$ and $M\geq 1$,

$$\limsup_{z\to\infty}\limsup_{n\to\infty}\mathbb{P}\Big[I_n(n-\lfloor\delta n\rfloor)\geq M\sigma\sqrt{\delta n}\Big|I_n\leq a_n(z)\Big]=\mathbb{P}[e_{1-\delta}\geq M\sqrt{\delta}], \tag{4.3.49}$$

which, by Chebyshev's inequality, is bounded by $\frac{\mathbb{E}[e_{1-\delta}]}{M\sqrt{\delta}}=\frac{4\sqrt{1-\delta}}{M\sqrt{2\pi}}$. Consequently,

$$\begin{aligned}&\limsup_{z\to\infty}\limsup_{n\to\infty}\mathbb{P}\Big(\sup_{0\leq k\leq\delta n}|I_n(n-k)-I_n|\geq\eta\sigma\sqrt{n}\Big|I_n\leq a_n(z)\Big) \qquad (4.3.50)\\ \leq\ & \frac{\varepsilon}{c_{13}}+\frac{2}{M}+\limsup_{z\to\infty}\limsup_{n\to\infty}\frac{\chi(\delta,z,n)}{\mathbb{P}(I_n\leq a_n(z))}.\end{aligned}$$

Let us estimate $\chi(\delta,z,n)$. One sees that

$$\chi(\delta,z,n)\ \leq\ \mathbb{E}\Big[\sum_{|u|=n}1_{\{u\in J_{z,L}^{1/2}(n);\ \sup_{0\leq k\leq\delta n}|V(u_{n-k})-V(u)|\geq\eta\sigma\sqrt{n};\ V(u_{n-\lfloor\delta n\rfloor})\leq M\sigma\sqrt{\delta n}\}}\Big].$$

By Lemma 4.2.4, it becomes that

$$\begin{aligned}\chi(\delta,z,n)\ &\leq\ \mathbb{E}\Big[e^{S_n};S_n\leq a_n(z),\underline{S}_n\geq -z,\underline{S}_{[n/2,n]}\geq a_n(z+L),\\ &\qquad S_{n-\lfloor\delta n\rfloor}\leq M\sigma\sqrt{\delta n},\sup_{0\leq k\leq\delta n}|S_{n-k}-S_n|\geq\eta\sigma\sqrt{n}\Big]\\ &\leq\ n^{3/2}e^{-z}\Upsilon(\delta,z,n),\end{aligned}$$

where $\Upsilon(\delta,z,n):=\mathbb{P}\Big(S_n\leq a_n(z),\ \underline{S}_n\geq -z,\ \underline{S}_{[n/2,n]}\geq a_n(z+L),\ S_{n-\lfloor\delta n\rfloor}\leq M\sigma\sqrt{\delta n}$, $\sup_{0\leq k\leq\delta n}|S_{n-k}-S_n|\geq\eta\sigma\sqrt{n},\ S_{n-\lfloor\delta n\rfloor}\leq M\sigma\sqrt{\delta n}\Big)$.

Reversing time yields that

$$\Upsilon(\delta,z,n) \le \mathbb{P}\Big(\underline{S}_n^- \ge -a_n(0), \underline{S}_{n/2}^- \ge -L, -S_n \in [-a_n(z), -a_n(z+L)],$$
$$\sup_{0\le k\le \delta n} |-S_k| \ge \eta\sigma\sqrt{n}, -S_{\lfloor \delta n\rfloor} \le M\sigma\sqrt{\delta n} - a_n(z+L)\Big). \quad (4.3.51)$$

Applying the Markov property at time $\lfloor \delta n\rfloor$, we obtain that

$$\Upsilon(\delta,z,n) = \mathbb{E}\Big[\Theta(-S_{\lfloor \delta n\rfloor}); \underline{S}_{\delta n}^- \ge -L, \sup_{0\le k\le \delta n} |-S_k| \ge \eta\sigma\sqrt{n}\Big], \quad (4.3.52)$$

where $\Theta(x) := 1_{\{x\le M\sigma\sqrt{\delta n} - a_n(z+L)\}}\mathbb{P}_x\Big(\underline{S}_{(1/2-\delta)n}^- \ge -L, \underline{S}_{(1-\delta)n}^- \ge -a_n(0), -S_{n-\lfloor \delta n\rfloor} \in [-a_n(z), -a_n(z+L)]\Big)$. Reversing time again implies that

$$\Theta(x) \le 1_{\{x\le M\sigma\sqrt{\delta n}\}}\mathbb{P}\Big(\underline{S}_{(1-\delta)n} \ge -z-L,$$
$$\underline{S}_{[n/2,(1-\delta)n]} \ge a_n(z+2L), S_{n-\lfloor \delta n\rfloor} \in [x+a_n(z+L), x+a_n(z)]\Big).$$

By (4.2.9), $\Theta(x) \le c_{15}(1+z+L)(1+L)(1+M\sigma\sqrt{\delta n}+2L)n^{-3/2}$. Plugging it into (4.3.52) and taking n large enough so that $1+2L < \eta\sigma\sqrt{\delta n}$, we get that

$$\Upsilon(\delta,z,n) \le c_{15}(1+z)(1+L)^2 n^{-3/2}(M+\eta)\sigma\sqrt{\delta n}\,\mathbb{E}\Big[\underline{S}_{\delta n}^- \ge -L, \sup_{0\le k\le \delta n} |-S_k| \ge \eta\sigma\sqrt{n}\Big].$$

Recall that $\chi(\delta,z,n) \le e^{-z}n^{3/2}\Upsilon(\delta,z,n)$. We check that

$$\chi(\delta,z,n) \le c_{15}e^{-z}(1+z)(1+L)^2(M+\eta)\sigma$$
$$\times \mathbb{E}_L\Big[\sup_{0\le k\le \delta n}(-S_k) \ge \eta\sigma\sqrt{n}\Big|\underline{S}_{\delta n}^- \ge 0\Big]\Big(\sqrt{\delta n}\,\mathbb{P}_L\Big[\underline{S}_{\delta n}^- \ge 0\Big]\Big). \quad (4.3.53)$$

On the one hand, by Theorem 1.1 of [53], $\mathbb{E}_L\Big[\sup_{0\le k\le \delta n}(-S_k) \ge \eta\sigma\sqrt{n}\Big|\underline{S}_{\delta n}^- \ge 0\Big]$ converges to $\mathbb{P}(\sup_{0\le s\le 1}\mathscr{M}_s \ge \eta/\sqrt{\delta})$ as $n\to\infty$. On the other hand, (4.2.6) shows that $\sqrt{\delta n}\,\mathbb{P}_L\Big[\underline{S}_{\delta n}^- \ge 0\Big]$ converges to $C_- R_-(L)$ as $n\to\infty$. Therefore,

$$\limsup_{n\to\infty}\chi(\delta,z,n) \le c_{15}e^{-z}(1+z)(1+L)^2(M+\eta)\sigma C_- R_-(L) \times \mathbb{P}(\sup_{0\le s\le 1}\mathscr{M}_s \ge \eta/\sqrt{\delta}).$$

Going back to (4.3.50) and letting $z\to\infty$, we deduce from Fact 4.3.7 that

$$\limsup_{z\to\infty}\limsup_{n\to\infty}\mathbb{P}\Big(\sup_{0\le k\le \delta n}|I_n(n-k) - I_n| \ge \eta\sigma\sqrt{n}\Big|I_n \le a_n(z)\Big)$$
$$\le \frac{\varepsilon}{c_{13}} + \frac{2}{M} + \frac{c_{15}(1+L)^2(M+\eta)\sigma C_- R_-(L) \times \mathbb{P}(\sup_{0\le s\le 1}\mathscr{M}_s \ge \eta/\sqrt{\delta})}{c_{13}}. \quad (4.3.54)$$

Notice that $\mathbb{P}(\sup_{0\le s\le 1}\mathscr{M}_s \ge \eta/\sqrt{\delta})$ decreases to 0 as $\delta \downarrow 0$. Take $M \ge 2/\varepsilon$. We conclude that for any $0<\varepsilon<c_{13}$,

$$\limsup_{\delta\to 0}\limsup_{z\to\infty}\limsup_{n\to\infty}\mathbb{P}\Big(\sup_{0\le k\le \delta n}|I_n(n-k)-I_n| \ge \eta\sigma\sqrt{n}\Big|I_n \le a_n(z)\Big) \le \frac{\varepsilon}{c_{13}}+\varepsilon, \qquad (4.3.55)$$

which completes the proof of Proposition 4.3.6. And Proposition 4.3.1 is thus proved. □

4.4 Proof of Theorem 4.1.1

Let us prove the main theorem now. It suffices to prove that for any continuous functional $F : D([0,1],\mathbb{R}) \to [0,1]$, we have

$$\lim_{n\to\infty}\Big|\mathbb{E}\Big[F\Big(\frac{I_n(\lfloor sn\rfloor)}{\sigma\sqrt{n}};0\le s\le 1\Big)\Big]-\mathbb{E}\Big[F(e_s,\, 0\le s\le 1)\Big]\Big| = 0. \qquad (4.4.1)$$

Proof of (4.4.1). Define for $A \ge 0$,

$$\mathscr{Z}[A] := \{u\in\mathbb{T} : V(u)\ge A > \max_{k<|u|}V(u_k)\}. \qquad (4.4.2)$$

For any particle $u\in\mathscr{Z}[A]$, there is a subtree rooted at u. If $|u|\le n$, let

$$I_n^u := \min_{v\ge u, |v|=n} V(v).$$

Moreover, assume m_n^u is the particle uniformly chosen in the set $\{|v| = n : v\ge u, V(v) = I_n^u\}$. Similarly, we write $[\![\varnothing, m_n^u]\!] := \{\varnothing =: m_0^u, m_1^u,\cdots, m_n^u\}$. The trajectory leading to m_n^u is denoted by $\{V(m_k^u); 0\le k\le n\}$. Let ω_A be the particle uniformly chosen in $\{u\in\mathscr{Z}[A] : |u|\le n, I_n^u = I_n\}$.

Let $\mathscr{Y}_A := \{\max_{u\in\mathscr{Z}[A]}|u|\le M, \max_{u\in\mathscr{Z}[A]}V(u)\le M\}$. Then for any $\varepsilon>0$, there exist $M := M(A,\varepsilon)$ large enough such that $\mathbb{P}(\mathscr{Y}_A^c)\le\varepsilon$. It follows that

$$\begin{aligned}&\Big|\mathbb{E}\Big[F\Big(\frac{I_n(\lfloor sn\rfloor)}{\sigma\sqrt{n}};0\le s\le 1\Big)\Big]-\mathbb{E}\Big[F\Big(\frac{I_n(\lfloor sn\rfloor)}{\sigma\sqrt{n}};0\le s\le 1\Big);\mathscr{Y}_A, |I_n-a_n(0)|\le A/2\Big]\Big|\\ \le\ & \varepsilon+\mathbb{P}[|I_n-a_n(0)|\ge A/2]. \qquad (4.4.3)\end{aligned}$$

We then check that for $n\ge M$,

$$\begin{aligned}&\mathbb{E}\Big[F\Big(\frac{I_n(\lfloor sn\rfloor)}{\sigma\sqrt{n}};0\le s\le 1\Big);\mathscr{Y}_A, |I_n-a_n(0)|\le A/2\Big] \qquad (4.4.4)\\ =\ & \mathbb{E}\Big[\sum_{u\in\mathscr{Z}[A]}1_{(u=\omega_A)}F\Big(\frac{V(m^u_{\lfloor sn\rfloor})}{\sigma\sqrt{n}};0\le s\le 1\Big);\mathscr{Y}_A, |I_n-a_n(0)|\le A/2\Big].\end{aligned}$$

Define another trajectory $\{\tilde{V}(m_k^u); 0 \le k \le n\}$ as follows.

$$\tilde{V}(m_k^u) := \begin{cases} V(u) & \text{if } k < |u|; \\ V(m_k^u) & \text{if } |u| \le k \le n. \end{cases} \tag{4.4.5}$$

It follows that

$$\begin{aligned} & \mathbb{E}\Big[F\Big(\frac{I_n(\lfloor sn \rfloor)}{\sigma\sqrt{n}}; 0 \le s \le 1\Big); \mathscr{Y}_A, |I_n - a_n(0)| \le A/2\Big] \qquad (4.4.6) \\ = \; & \mathbb{E}\Big[\sum_{u \in \mathscr{Z}[A]} 1_{(u=\omega_A)} F\Big(\frac{\tilde{V}(m^u_{\lfloor sn \rfloor})}{\sigma\sqrt{n}}; 0 \le s \le 1\Big); \mathscr{Y}_A, |I_n - a_n(0)| \le A/2\Big] + o_n(1), \end{aligned}$$

where $o_n(1) \to 0$ as n goes to infinity.

Define the sigma-field $\mathscr{G}_A := \sigma\{(u, V(u), I_n^u); u \in \mathscr{Z}[A]\}$. Note that on $\mathscr{Y}_A$, $I_n = \min_{u \in \mathscr{Z}[A]} I_n^u$ as long as $n \ge M$. One sees that $\mathscr{Y}_A \cap \{|I_n - a_n(0)| \le A/2\}$ is $\mathscr{G}_A$-measurable for all n large enough. Thus,

$$\begin{aligned} & \mathbb{E}\Big[\sum_{u \in \mathscr{Z}[A]} 1_{(u=\omega_A)} F\Big(\frac{\tilde{V}(m^u_{\lfloor sn \rfloor})}{\sigma\sqrt{n}}; 0 \le s \le 1\Big); \mathscr{Y}_A, |I_n - a_n(0)| \le A/2\Big] \qquad (4.4.7) \\ = \; & \mathbb{E}\Big[\sum_{u \in \mathscr{Z}[A]} 1_{(u=\omega_A)} \mathbb{E}\Big[F\Big(\frac{\tilde{V}(m^u_{\lfloor sn \rfloor})}{\sigma\sqrt{n}}; 0 \le s \le 1\Big)\Big| \mathscr{G}_A, u = \omega_A\Big]; \mathscr{Y}_A, |I_n - a_n(0)| \le A/2\Big]. \end{aligned}$$

Further, we notice by the branching property that conditioned on $\{(u, V(u)); u \in \mathscr{Z}[A]\}$, the subtrees generated by $u \in \mathscr{Z}[A]$ are independent copies of the original one, started from $V(u)$, respectively. Therefore, given $\mathscr{Y}_A \cap \{|I_n - a_n(0)| \le A/2\}$,

$$\begin{aligned} & 1_{(u=\omega_A)} \mathbb{E}\Big[F\Big(\frac{\tilde{V}(m^u_{\lfloor sn \rfloor})}{\sigma\sqrt{n}}; 0 \le s \le 1\Big)\Big| \mathscr{G}_A, u = \omega_A\Big] \\ = \; & 1_{(u=\omega_A)} \mathbb{E}\Big[F\Big(\frac{I(\lfloor s(n-|u|) \rfloor)}{\sigma\sqrt{n-|u|}}; 0 \le s \le 1\Big)\Big| I_{n-|u|} \le a_n(-r_u)\Big] + o_n(1), \end{aligned}$$

where $r_u := \min\{\min_{v \in \mathscr{Z}[A] \setminus \{u\}} I_n^v - a_n(0), A/2\} - V(u)$ is independent of $I_{n-|u|}$. Thus, (4.4.6) becomes that

$$\begin{aligned} & \mathbb{E}\Big[F\Big(\frac{I_n(\lfloor sn \rfloor)}{\sigma\sqrt{n}}; 0 \le s \le 1\Big); \mathscr{Y}_A, |I_n - a_n(0)| \le A/2\Big] \qquad (4.4.8) \\ = \; & \mathbb{E}\Big[\sum_{u \in \mathscr{Z}[A]} 1_{(u=\omega_A)} \mathbb{E}\Big[F\Big(\frac{I(\lfloor s(n-|u|) \rfloor)}{\sigma\sqrt{n-|u|}}; 0 \le s \le 1\Big)\Big| I_{n-|u|} \le a_n(-r_u)\Big]; \\ & \quad \mathscr{Y}_A, |I_n - a_n(0)| \le A/2\Big] + o_n(1). \end{aligned}$$

The event $\mathscr{Y}_A \cap \{|I_n - a_n(0)| \le A/2\}$ ensures that $A/2 + M \ge -r_u \ge A/2$. The conditioned convergence has been given in Proposition 4.3.1. We need a slightly stronger version here.

According to Proposition 4.3.1, for any $\varepsilon > 0$, there exists $z_\varepsilon > 0$ such that for all $z \ge z_\varepsilon$,

$$\limsup_{n\to\infty} \Big|\mathbb{E}\Big[F\Big(\frac{I_n(\lfloor sn \rfloor)}{\sigma\sqrt{n}}; 0 \le s \le 1\Big)\Big| I_n \le a_n(z)\Big] - \mathbb{E}[F(e_s, 0 \le s \le 1)]\Big| < \varepsilon. \quad (4.4.9)$$

Thus, for any $z \ge z_\varepsilon$, there exists $N_z \ge 1$ such that for any $n \ge N_z$,

$$\Big|\mathbb{E}\Big[F\Big(\frac{I_n(\lfloor sn \rfloor)}{\sigma\sqrt{n}}; 0 \le s \le 1\Big)\Big| I_n \le a_n(z)\Big] - \mathbb{E}[F(e_s, 0 \le s \le 1)]\Big| < 2\varepsilon. \quad (4.4.10)$$

Take $A = 2z_\varepsilon$ and $K = M$. We say that for n sufficiently large,

$$\sup_{z\in[z_\varepsilon, z_\varepsilon+K]} \Big|\mathbb{E}\Big[F\Big(\frac{I(\lfloor s(n) \rfloor)}{\sigma\sqrt{n}}; 0 \le s \le 1\Big)\Big| I_n \le a_n(z)\Big] - \mathbb{E}[F(e_s, 0 \le s \le 1)]\Big| \le 3\varepsilon. \quad (4.4.11)$$

In the lattice case, (4.4.11) follows immediately. We only need to prove it in the non-lattice case.

Recall that $\Sigma_n(F,z) = \mathbb{E}\Big[F\Big(\frac{I_n(\lfloor sn \rfloor)}{\sigma\sqrt{n}}; 0 \le s \le 1\Big);\ I_n \le a_n(z)\Big]$ with $0 \le F \le 1$. Then, for any $\ell > 0$ and $z \ge 0$,

$$\begin{aligned} & \Big|\frac{\Sigma_n(F,z)}{\Sigma_n(1,z)} - \frac{\Sigma_n(F,z+\ell)}{\Sigma_n(1,z+\ell)}\Big| && (4.4.12) \\ \le\ & \Big|\frac{\Sigma_n(F,z) - \Sigma_n(F,z+\ell)}{\Sigma_n(1,z)}\Big| + \Big|\frac{\Sigma_n(F,z+\ell)}{\Sigma_n(1,z)} - \frac{\Sigma_n(F,z+\ell)}{\Sigma_n(1,z+\ell)}\Big| \\ =\ & \frac{1}{\Sigma_n(1,z)}\Big(\Big|\Sigma_n(F,z) - \Sigma_n(F,z+\ell)\Big| + \frac{\Sigma_n(F,z+\ell)}{\Sigma_n(1,z+\ell)}\Big|\Sigma_n(1,z+\ell) - \Sigma_n(1,z)\Big|\Big). \end{aligned}$$

Since $0 \le F \le 1$, the two following inequalities

$$\begin{aligned} \Big|\Sigma_n(F,z) - \Sigma_n(F,z+\ell)\Big| &= \mathbb{E}\Big[F\Big(\frac{I_n(\lfloor sn \rfloor)}{\sigma\sqrt{n}}; 0 \le s \le 1\Big);\ a_n(z+\ell) < I_n \le a_n(z)\Big] \\ &\le \mathbb{P}(a_n(z+\ell) < I_n \le a_n(z)), \end{aligned}$$

and $\frac{\Sigma_n(F,z+\ell)}{\Sigma_n(1,z+\ell)} \le 1$ hold. Note also that $|\Sigma_n(1,z+\ell) - \Sigma_n(1,z)| = \mathbb{P}(a_n(z+\ell) < I_n \le a_n(z))$. It follows that

$$\begin{aligned} \Big|\frac{\Sigma_n(F,z)}{\Sigma_n(1,z)} - \frac{\Sigma_n(F,z+\ell)}{\Sigma_n(1,z+\ell)}\Big| &\le 2\frac{\mathbb{P}(a_n(z+\ell) < I_n \le a_n(z))}{\mathbb{P}(I_n \le a_n(z))} && (4.4.13) \\ &= 2 - 2\frac{\mathbb{P}(I_n \le a_n(z+\ell))}{\mathbb{P}(I_n \le a_n(z))}. \end{aligned}$$

In view of Fact 4.3.4, we take $\frac{3}{2}\log n - \Lambda_{\varepsilon'} \ge \ell + z > z \ge \Lambda_{\varepsilon'}$ so that for any $n \ge N_{\varepsilon'}$,

$$\frac{\mathbb{P}(I_n \le a_n(z+\ell))}{\mathbb{P}(I_n \le a_n(z))} \ge \frac{(C-\varepsilon')(z+\ell)e^{-z-\ell}}{(C+\varepsilon')ze^{-z}} \ge \frac{C-\varepsilon'}{C+\varepsilon'}e^{-\ell}. \quad (4.4.14)$$

For $\varepsilon' = C\varepsilon/8 > 0$, we choose $\zeta = \frac{\varepsilon}{4}$ so that $\frac{C-\varepsilon'}{C+\varepsilon'}e^{-\zeta} \geq 1 - \frac{\varepsilon}{2}$. As a consequence, for any $\Lambda_{\varepsilon'} \leq z \leq \frac{3}{2}\log n - \Lambda_{\varepsilon'} - \zeta$, $0 \leq \ell \leq \zeta$ and $n \geq N_{\varepsilon'}$,

$$\Big|\frac{\Sigma_n(F,z)}{\Sigma_n(1,z)} - \frac{\Sigma_n(F,z+\ell)}{\Sigma_n(1,z+\ell)}\Big| \leq 2\Big(1 - \frac{C-\varepsilon'}{C+\varepsilon'}e^{-\ell}\Big) \leq \varepsilon. \tag{4.4.15}$$

For $\varepsilon > 0$, z_ε can be chosen so that $[z_\varepsilon, z_\varepsilon + K] \subset [\Lambda_{\varepsilon'}, \frac{3}{2}\log n - \Lambda_{\varepsilon'}]$ for $n \geq e^K N_{\varepsilon'}$. For any integer $0 \leq j \leq \lceil K/\zeta \rceil$, let $z_j := z_\varepsilon + j\zeta$. Then $[z_\varepsilon, z_\varepsilon + K] \subset \cup_{0\leq j\leq\lceil K/\zeta\rceil}[z_j, z_{j+1}]$. Take $N'_\varepsilon = \max_{0\leq j\leq\lceil K/\zeta\rceil}\{N_{z_j}, e^K N_{\varepsilon'}\}$. By (4.4.10) and (4.4.15), we conclude that for any $n \geq N'_\varepsilon$,

$$\begin{aligned}
& \sup_{z\in[z_\varepsilon, z_\varepsilon+K]} \Big|\mathbb{E}\Big[F\Big(\frac{I_n(\lfloor sn\rfloor)}{\sigma\sqrt{n}}; 0 \leq s \leq 1\Big)\Big| I_n \leq a_n(z)\Big] - \mathbb{E}[F(e_s, 0 \leq s \leq 1)]\Big| \\
\leq\ & \sup_{0\leq j\leq\lceil K/\zeta\rceil} \Big|\frac{\Sigma_n(F,z_j)}{\Sigma_n(1,z_j)} - \mathbb{E}[F(e_s,\ 0 \leq s \leq 1)]\Big| + \sup_{0\leq j<\lceil K/\zeta\rceil}\ \sup_{z_j\leq z\leq z_{j+1}} \Big|\frac{\Sigma_n(F,z)}{\Sigma_n(1,z)} - \frac{\Sigma_n(F,z_j)}{\Sigma_n(1,z_j)}\Big| \\
\leq\ & 3\varepsilon.
\end{aligned}$$

We continue to prove the main theorem. Since $\sum_{u\in\mathscr{X}[A]} 1_{(u=\omega_A)} = 1$, we deduce from (4.4.8) and (4.4.11) that for n sufficiently large,

$$\begin{aligned}
& \Big|\mathbb{E}\Big[F\Big(\frac{I_n(\lfloor sn\rfloor)}{\sigma\sqrt{n}}; 0 \leq s \leq 1\Big); \mathscr{Y}_A, |I_n - a_n(0)| \leq A/2\Big] - \mathbb{E}[F(e_s,\ 0 \leq s \leq 1)]\Big| \\
\leq\ & 3\varepsilon\mathbb{P}(\mathscr{Y}_A; |I_n - a_n(0)| \leq A/2) + o_n(1) + \mathbb{P}(\mathscr{Y}_A^c) + \mathbb{P}(|I_n - a_n(0)| \geq A/2) \\
\leq\ & 4\varepsilon + o_n(1) + \mathbb{P}(|I_n - a_n(0)| \geq A/2).
\end{aligned}$$

Going back to (4.4.3), we conclude that for n large enough,

$$\Big|\mathbb{E}\Big[F\Big(\frac{I_n(\lfloor sn\rfloor)}{\sigma\sqrt{n}}; 0 \leq s \leq 1\Big)\Big] - \mathbb{E}[F(e_s,\ 0 \leq s \leq 1)]\Big| \leq 5\varepsilon + 2\mathbb{P}(|I_n - a_n(0)| \geq A/2) + o_n(1).$$

Let n go to infinity and then make $\varepsilon \downarrow 0$. Therefore,

$$\begin{aligned}
& \limsup_{n\to\infty} \Big|\mathbb{E}\Big[F\Big(\frac{I_n(\lfloor sn\rfloor)}{\sigma\sqrt{n}}; 0 \leq s \leq 1\Big)\Big] - \mathbb{E}[F(e_s,\ 0 \leq s \leq 1)]\Big| \\
\leq\ & \limsup_{z\to\infty}\limsup_{n\to\infty} 2\mathbb{P}(|I_n - a_n(0)| \geq z).
\end{aligned} \tag{4.4.16}$$

It remains to show that $\limsup_{z\to\infty}\limsup_{n\to\infty}\mathbb{P}(|I_n - a_n(0)| \geq z) = 0$. Because of Fact (4.3.7), it suffices to prove that

$$\limsup_{z\to\infty}\limsup_{n\to\infty}\mathbb{P}(I_n \geq a_n(0) + z) = 0. \tag{4.4.17}$$

In the non-lattice case, Theorem 1.1 of [4] implies it directly. In the lattice case, we see that for n large enough,

$$\mathbb{P}(I_n \geq a_n(0)+z) \leq \mathbb{E}\Big[\prod_{u\in\mathscr{Z}[A]}(1-\Phi_u(z,n));\mathscr{Y}_A\Big]+\varepsilon, \tag{4.4.18}$$

with $\Phi_u(z,n) := \mathbb{P}(I_{n-|u|} \leq a_n(V(u)-z))$. Take $A = 2z$ here. Then it follows from Fact 4.3.7 that for n large enough and for any particle $u \in \mathscr{Z}[A]$,

$$\Phi_u(z,n) \geq c_{13}/2(V(u)-z)e^{z-V(u)} \geq \frac{c_{13}}{4}V(u)e^{z-V(u)}. \tag{4.4.19}$$

(4.4.18) hence becomes that

$$\begin{aligned}\limsup_{n\to\infty}\mathbb{P}(I_n \geq a_n(0)+z) &\leq \mathbb{E}\Big[\prod_{u\in\mathscr{Z}[A]}(1-\frac{c_{13}}{4}V(u)e^{z-V(u)});\mathscr{Y}_A\Big]+\varepsilon \\ &\leq \mathbb{E}\Big[\exp\Big(-\frac{c_{13}}{4}e^{z}\sum_{u\in\mathscr{Z}[A]}V(u)e^{-V(u)}\Big)\Big]+\varepsilon.\end{aligned}$$

It has been proved that as A goes to infinity, $\sum_{u\in\mathscr{Z}[A]}V(u)e^{-V(u)}$ converges almost surely to some limit D_∞, which is strictly positive on the set of non-extinction of $\mathbb{T}$, (see (5.2) in [4]). We end up with

$$\limsup_{z\to\infty}\limsup_{n\to\infty}\mathbb{P}(I_n \geq a_n(0)+z) \leq \varepsilon, \tag{4.4.20}$$

which completes the proof of Theorem 4.1.1. □

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Chapitre 5

Increasing paths on N-ary trees

Summary. We consider a rooted N-ary tree. To every vertex of this tree, we attach an i.i.d. continuous random variable. A vertex is called accessible if along its ancestral line, the attached random variables are increasing. We keep accessible vertices and kill all the others. For any positive constant α, we describe the asymptotic behaviors of the population at the $\lfloor \alpha N \rfloor$-th generation as N goes to infinity.

Keywords. Increasing path ; House of Cards.

5.1 Introduction

5.1.1 The model

We consider an N-ary tree $T^{(N)}$, which is rooted at $\varnothing$, so that each vertex in $T^{(N)}$ has exactly N children. To every vertex $\sigma \in T^{(N)}$, we assign a continuous random variable, denoted by x_σ. All these variables x_σ, $\sigma \in T^{(N)}$ are i.i.d. Let $|\sigma|$ denote the generation of σ, and σ_i (for $0 \le i \le |\sigma|$) denote its ancestor at generation i. The ancestral line of σ is denoted by

$$[\![\varnothing, \sigma]\!] := \{\sigma_0 := \varnothing, \sigma_1, \cdots, \sigma_{|\sigma|} := \sigma\},$$

which is also the unique shortest path relating σ to the root $\varnothing$. A vertex σ is called accessible if along its ancestral line, the assigned random variables are increasing, i.e.,

$$\sigma \text{ accessible } \Leftrightarrow x_\varnothing < x_{\sigma_1} < \cdots < x_\sigma. \tag{5.1.1}$$

This model is called accessibility percolation by Nowak and Krug [123]. We also call $[\![\varnothing, \sigma]\!]$ an accessible path if σ is accessible.

The model comes from evolutionary biology, in which both mutation and selection involve. As the main source of evolutionary novelty, mutations act on the genetic constitution of an organism. In our setting, each vertex represents one gene type, or genotype. A certain genotype may reproduce several new genotypes through mutations. The mechanism of successive mutations hence gives the structure of trees if we also assume that each mutation gives rise to a new genotype. Selection involves so that organisms better adapted to their respective surroundings are favored to survive. We suppose that each genotype (vertex) has an associated fitness value, which is represented by the assigned random variable. In the strong-selection/weak mutation regime, we assume that only mutations which give rise to a larger fitness value survive. In this way, the survival mutational pathways are noted by the accessible vertices. In this paper, we use 'House of Cards' model (see [96]), in which all fitness values are i.i.d. As is explained in [69], it serves as a null model.

A variation of our model by replacing N-ary trees with N-dimensional hypercube has been considered in [30] and [80]. More models are introduced in [11] [69] to explain evolution via mutation and selection.

5.1.2 Main results

For any $k \geq 1$, let $\mathscr{A}_{N,k} := \{\sigma \in T^{(N)} : |\sigma| = k, \sigma \text{ is accessible}\}$. We define

$$Z_{N,k} := \sum_{|\sigma|=k} 1_{(\sigma \in \mathscr{A}_{N,k})}, \quad \forall k \geq 1. \tag{5.1.2}$$

Let $\alpha > 0$. For convenience, we let αN represent the integer $\lfloor \alpha N \rfloor$ throughout this paper. We are interested in the behavior of $Z_{N,\alpha N}$. Since we are only concerned with the order of the random variables, under the assumption of continuity of their law, changing the precise distribution will not influence the results. Without loss of generality, we assume throughout the paper that the assigned random variables are distributed uniformly in $[0,1]$, i.e., $\forall \sigma \in T^{(N)}$, x_σ has the uniform distribution in $[0,1]$, which is denoted by $U[0,1]$.

For any $x \in [0,1]$, we introduce the following probability measure :

$$\mathbb{P}_x(\cdot) := \mathbb{P}(\cdot | x_\varnothing = x). \tag{5.1.3}$$

We now state the main results.

Theorem 5.1.1. *There exists a phase transition at $\alpha = e$ under $\mathbb{P}_0$.*

(1) For $\alpha \in (0,e)$,

$$\lim_{N\to\infty} \mathbb{P}_0\left[Z_{N,\alpha N} \geq 1\right] = 1. \tag{5.1.4}$$

(2) For $\alpha \geq e$,

$$\lim_{N\to\infty} \mathbb{P}_0\left[Z_{N,\alpha N} \geq 1\right] = 0. \tag{5.1.5}$$

Remark 5.1.2. *Nowak and Krug [123] showed $\mathbb{P}_0[Z_{N,\alpha N} \geq 1] > 0$ when $0 < \alpha < 1$. This theorem is also proven independently by Roberts and Zhao [129] by considering some typical increasing paths. However, in this paper, we apply a coupling between this model and certain branching process which comes from an accessibility percolation based on a Galton-Watson tree. This coupling may have independent interest.*

Theorem 5.1.1 tells us that, for N large, roughly speaking, the population of accessible vertices survives until the eN-th generation and then dies out. Let us describe the asymptotic behaviors of the population more precisely by the following theorem.

Theorem 5.1.3. *Let $\theta(\alpha) := \alpha(1-\log\alpha)$ for $\alpha > 0$.*

(i) When $\alpha \in (0,e)$, the following convergence holds $\mathbb{P}_0-$almost surely,

$$\lim_{N\to\infty} \frac{Z_{N,\alpha N}}{N} = \theta(\alpha) > 0. \tag{5.1.6}$$

(ii) When $\alpha = e$, we have

$$\mathbb{P}_0\left[Z_{N,\alpha N} \geq 1\right] = N^{-3/2+o_N(1)} \text{ as } N \to \infty, \tag{5.1.7}$$

where $\{o_N(1)\}_{N\geq 1}$ is a sequence of real numbers which goes to zero as $N \to \infty$.

(iii) When $\alpha > e$, we have

$$\lim_{N\to\infty} \frac{\log \mathbb{P}_0\left[Z_{N,\alpha N} \geq 1\right]}{N} = \theta(\alpha) < 0. \tag{5.1.8}$$

This is the principal result in this paper, which gives us a clear picture of the evolution of the accessible population. In the proof, we generalize the idea in [129] about typical increasing paths and get more information about the population at the critical generation.

The figure of the limit function $\theta(\alpha)$ is shown in FIGURE 5.1 at the end of this chapter.

One sees that the maximum of $\alpha \to \theta(\alpha)$ is reached at $\alpha = 1$. We turn to consider $Z_{N,\alpha N}$ when $\alpha = 1$. Let $\mathscr{L}(X,\mathbb{P}_x)$ denote the law of random variable X under $\mathbb{P}_x$. The theorem is given as follows.

Theorem 5.1.4. *Let $\lambda > 0$ fixed. Then the following convergence in law holds as $N \to \infty$:*

$$\mathscr{L}\left(\frac{Z_{N,N}}{m_N};\mathbb{P}_{\frac{\lambda}{N}}\right) \to e^{-\lambda} \times W, \tag{5.1.9}$$

where W is an exponential variable with mean 1 and $m_N := \frac{N^N}{N!}$.

Remark 5.1.5. *A similar result to Theorem 5.1.4 has been given in [30] by considering the accessible paths in the N-dimensional hypercubes. Our proof is mainly inspired by it.*

It is possible to replace the N-ary tree by the Galton-Watson tree whose offspring is Poisson with parameter N, in which case all these results still hold.

The rest of the paper is organized as follows. In Section 5.2, we prove Theorem 5.1.1 by comparing the model with certain branching process. In Section 5.3 we consider several typical increasing paths and prove Theorem 5.1.3. Finally, in Section 5.4, we prove Theorem 5.1.4.

Throughout the paper, we use the letter c with subscript to denote a finite and positive constant.

5.2 Phase transition at $\alpha = e$

5.2.1 The generating function of $Z_{N,k}$

Note that for any $N, k \geq 1$, $Z_{N,k} = \sum_{|\sigma|=k} 1_{(\sigma \in \mathscr{A}_{N,k})} = \sum_{|\sigma|=k} 1_{(x_\varnothing < x_{\sigma_1} < \cdots < x_\sigma)}$. We observe that

$$\mathbb{E}\Big[Z_{N,k}\Big] = N^k \mathbb{P}(x_\varnothing < x_{\sigma_1} < \cdots < x_\sigma) = \frac{N^k}{(k+1)!}, \tag{5.2.1}$$

since $x_\varnothing, x_{\sigma_1}, x_{\sigma_2} \cdots, x_\sigma$ are i.i.d. and distributed uniformly in $[0,1]$. Moreover,

$$\mathbb{E}_x\Big[Z_{N,k}\Big] = N^k \mathbb{P}_x(x < x_{\sigma_1} < \cdots < x_\sigma \leq 1) = N^k \frac{(1-x)^k}{k!}, \quad \forall x \in [0,1]. \tag{5.2.2}$$

More generally, for any $0 \leq a < b \leq 1$, we define $Z_{N,k}(a,b)$ as follows :

$$Z_{N,k}(a,b) := \sum_{|\sigma|=k} 1_{(a < x_{\sigma_1} < \cdots < x_{\sigma_k} \leq b)}, \quad \forall k \geq 1. \tag{5.2.3}$$

For convenience, we write $Z_{N,k}(b)$ for $Z_{N,k}(0,b)$ and set $Z_{N,0}(b) \equiv 1$. One sees that $Z_{N,k}(b-a)$ and $Z_{N,k}(a,b)$ have the same law. Let $f_k^{(N)}(s,b)$ be the generating function of $Z_{N,k}(b)$, i.e.,

$$f_k^{(N)}(s,b) := \mathbb{E}\Big[s^{Z_{N,k}(b)}\Big], \quad \forall s \in [0,1]. \tag{5.2.4}$$

For $k=1$, $Z_{N,1}(b)$ is a binomial variable with parameter (N,b). So $f_1^{(N)}(s,b)=[1-b+sb]^N$.

For any $k\geq 1$, one observes that

$$Z_{N,k+1}(b)=\sum_{|\sigma|=1}1_{(x_\sigma<b)}\sum_{|\omega|=k+1}1_{(\omega_1=\sigma)}1_{(x_\sigma<x_{\omega_2}<\cdots<x_\omega\leq b)}. \tag{5.2.5}$$

For all vertices σ of the first generation, the variables $1_{(x_\sigma<b)}\sum_{|\omega|=k+1}1_{(\omega_1=\sigma)}1_{(x_\sigma<x_{\omega_2}<\cdots<x_\omega\leq b)}$ are i.i.d., and given $\{x_\sigma=y<b\}$, $\sum_{|\omega|=k+1}1_{(\omega_1=\sigma)}1_{(x_\sigma<x_{\omega_2}<\cdots<x_\omega\leq b)}$ is distributed as $Z_{N,k}(y,b)$. It follows that for any $k\geq 1$ and any $b\in[0,1]$,

$$\begin{aligned}
f_{k+1}^{(N)}(s,b) &= \left[1-b+\int_0^b dy\mathbb{E}\left(s^{Z_{N,k}(y,b)}\right)\right]^N\\
&= \left[1-b+\int_0^b f_k^{(N)}(s,b-y)dy\right]^N\\
&= \left[1-b+\int_0^b f_k^{(N)}(s,y)dy\right]^N.
\end{aligned} \tag{5.2.6}$$

For brevity, we denote the generating function of $Z_{N,k}$ under $\mathbb{P}_0$ by $f_k^{(N)}(s)$ instead of $f_k^{(N)}(s,1)$. Immediately,

$$f_{k+1}^{(N)}(s)=\left[\int_0^1 f_k^{(N)}(s,y)dy\right]^N,\quad \forall k\geq 1. \tag{5.2.7}$$

This gives that

$$\mathbb{P}_0[Z_{N,k}\geq 1]=1-f_k^{(N)}(0)=1-\left[\int_0^1 f_{k-1}^{(N)}(0,y)dy\right]^N,\quad \forall k\geq 2. \tag{5.2.8}$$

To study the law of $Z_{N,k}$, it suffices to study (5.2.6). However, it is quite difficult to investigate analytically the sequence $f_k^{(N)}$, $k\geq 1$, from the recursive relation (5.2.6). We thus turn to study the accessible vertices via their paths. By controlling accessible paths, we get certain sub-trees which are actually Galton-Watson. In this way, a lower bound for the accessible population is obtained.

5.2.2 Coupling with a branching process

In the same probability space, we introduce accessibility percolation on a Galton-Watson tree as follows. For $\Lambda>0$, let $\mathscr{T}^\Lambda$ be a Galton-Watson tree rooted also at $\varnothing$, whose offspring distribution is Poisson with parameter Λ. To each vertex $\xi\in\mathscr{T}^\Lambda\setminus\{\varnothing\}$, we attach an random variable x_ξ, which is independent of $x_\varnothing$. Assume that all these variables x_ξ, $\xi\in\mathscr{T}^\Lambda$ are i.i.d., following the law $U[0,1]$. Similarly, let $[\![\varnothing,\xi]\!]$ denote the ancestral line of ξ in $\mathscr{T}^\Lambda$. We keep ξ if the attached random variables along its ancestral line $[\![\varnothing,\xi]\!]$ is *decreasing* and delete all other vertices. Let $D_k^{(\Lambda)}$

be the number of individuals alive at k-th generation. Let $d_k^{(\Lambda)}(s,x)$ denote the generating function of $D_k^{(\Lambda)}$ under $\mathbb{P}_x$. Similarly to (5.2.6), we get the following recursive equation.

$$d_{k+1}^{(\Lambda)}(s,x) = \mathbb{E}_x\Big[s^{D_{k+1}^{(\Lambda)}}\Big] = \exp\Big\{-\Lambda x + \Lambda\int_0^x d_k^{(\Lambda)}(s,y)dy\Big\}, \quad \forall k \geq 1. \tag{5.2.9}$$

In particular, $d_1^{(\Lambda)}(s,x) = \exp\{\Lambda x(s-1)\}$. We also note that $d_k^{(\Lambda)}(s,x) \leq d_k^{(\Lambda)}(s,y)$ if $x \geq y$.

We compare the generating functions $f_k^{(N)}$ and $d_k^{(\Lambda)}$ via the following lemma.

Lemma 5.2.1. *For any $0 < \Lambda \leq N$ and $u \in [0,1]$, we have*

$$f_k^{(N)}(s, \frac{\Lambda}{N}u) \leq d_k^{(\Lambda)}(s,u), \quad \forall k \geq 1. \tag{5.2.10}$$

Proof. For $N \geq \Lambda > 0$ and $u \in [0,1]$,

$$f_1^{(N)}(s, \frac{\Lambda}{N}u) = \Big(1 - \frac{\Lambda}{N}u + \frac{\Lambda}{N}us\Big)^N \leq \exp\{\Lambda u(s-1)\} = d_1^{(\Lambda)}(s,u). \tag{5.2.11}$$

Assume that $f_k^{(N)}(s, \frac{\Lambda}{N}u) \leq d_k^{(\Lambda)}(s,u)$ holds for $k \geq 1$. Then,

$$\begin{aligned} f_{k+1}^{(N)}(s, \frac{\Lambda}{N}u) &= \Big[1 - \frac{\Lambda}{N}u + \int_0^{\Lambda u/N} f_k^{(N)}(s,y)dy\Big]^N \\ &= \Big[1 - \frac{\Lambda}{N}u + \frac{\Lambda}{N}\int_0^u f_k^{(N)}(s, \frac{\Lambda}{N}v)dv\Big]^N \\ &\leq \exp\Big\{-\Lambda u + \Lambda\int_0^u f_k^{(N)}(s, \frac{\Lambda}{N}v)dv\Big\}, \end{aligned}$$

which is bounded by $\exp\Big\{-\Lambda u + \Lambda\int_0^u d_k^{(\Lambda)}(s,v)dv\Big\}$. It follows from (5.2.9) that

$$f_{k+1}^{(N)}(s, \frac{\Lambda}{N}u) \leq d_{k+1}^{(\Lambda)}(s,u). \tag{5.2.12}$$

Therefore, by induction on k, we have $f_k^{(N)}(s, \frac{\Lambda}{N}u) \leq d_k^{(\Lambda)}(s,u)$ for any $k \geq 1$. $\square$

With the help of this lemma, we show that with positive probability, there exists at least one accessible vertex at the αN-th generation for $\alpha < e$.

Lemma 5.2.2. *Let $\alpha \in (0,e)$. For any $\delta \in (\frac{\alpha}{e}, 1 \wedge \alpha)$, there exists some positive constant $c(\delta,\alpha) > 0$ such that*

$$\inf_{N \geq 1} \mathbb{P}\Big[Z_{N,\alpha N}(\delta) \geq 1\Big] > c(\delta,\alpha). \tag{5.2.13}$$

Proof. Set $K = \alpha N$. It follows from (5.2.10) that for $a \in \mathbb{N}_+$ and $a \leq N$,

$$f_a^{(N)}(s, \frac{a\delta}{K}) \leq d_a^{(a)}(s, \frac{\delta N}{K}) \leq d_a^{(a)}(s, \frac{\delta}{\alpha}). \tag{5.2.14}$$

For convenience, we write $h(s) = h_{a,N,K,\delta}(s) := f_a^{(N)}(s, \frac{a\delta}{K})$ and $\widehat{d}(s) = \widehat{d}_{a,\delta,\alpha}(s) := d_a^{(a)}(s, \frac{\delta}{\alpha})$, both of which are generating functions, satisfying $h(s) \leq \widehat{d}(s)$ for $N \geq a$.

Let $J \geq 0$ and $\kappa \in \{0, 1, \cdots, a-1\}$ be such that $K = aJ + \kappa$. Let $\mathscr{B}_K^{(N)}(\delta)$ be the collection of vertices σ in $T^{(N)}$ such that

$$\frac{\kappa + aj}{K}\delta < x_{\sigma_{\kappa+aj+1}} < \cdots < x_{\sigma_{\kappa+aj+j}} \leq \frac{\kappa + aj + a}{K}\delta, \quad \forall j \in \{0, \cdots, J-1\}, \tag{5.2.15}$$

and that

$$0 < x_{\sigma_1} < \cdots < x_{\sigma_\kappa} \leq \frac{\kappa}{K}\delta. \tag{5.2.16}$$

where $K := |\sigma|$. According to the definition of $\mathscr{B}_K^{(N)}(\delta)$, one sees that

$$Z_{N,K}(\delta) \geq \#\mathscr{B}_K^{(N)}(\delta) = \sum_{|\omega|=\kappa} 1_{(0<x_{\omega_1}<\cdots<x_\omega \leq \frac{\kappa}{K}\delta)} \sum_{|\sigma|=K} 1_{(\sigma_\kappa=\omega)} 1_{(\sigma \in \mathscr{B}_K^{(N)})}, \tag{5.2.17}$$

where given $\{0 < x_{\omega_1} < \cdots < x_\omega \leq \frac{\kappa}{K}\delta\}$, the generating function of $\sum_{|\sigma|=K} 1_{(\sigma_\kappa=\omega)} 1_{(\sigma \in \mathscr{B}_K^{(N)})}$ is $\underbrace{h \circ \cdots \circ h}_{J} =: h^{\circ J}$. As a consequence,

$$\begin{aligned}
& \mathbb{P}_0\Big[Z_{N,K}(\delta) \geq 1\Big] \geq \mathbb{P}_0\Big[\#\mathscr{B}_K^{(N)}(\delta) \geq 1\Big] \\
\geq \; & \mathbb{P}_0\Big[\sum_{|\omega|=\kappa} 1_{(0<x_{\omega_1}<\cdots<x_\omega \leq \frac{\kappa}{K}\delta)} \geq 1\Big] \mathbb{P}_0\Big[\sum_{|\sigma|=K} 1_{(\sigma_\kappa=\omega)} 1_{(\sigma \in \mathscr{B}_K^{(N)})} \geq 1 \Big| 0 < x_{\omega_1} < \cdots < x_\omega \leq \frac{\kappa}{K}\delta\Big] \\
= \; & \Big(1 - f_\kappa^{(N)}(0, \frac{\kappa}{K}\delta)\Big)\Big(1 - h^{\circ J}(0)\Big),
\end{aligned}$$

since the generating function of $Z_{N,\kappa}(\frac{\kappa}{K}\delta) = \sum_{|\omega|=\kappa} 1_{(0<x_{\omega_1}<\cdots<x_\omega \leq \frac{\kappa}{K}\delta)}$ is $f_\kappa^{(N)}(s, \frac{\kappa}{K}\delta)$. Applying the inequality (5.2.14) to $f_\kappa^{(N)}(0, \frac{\kappa}{K}\delta)$ and h, respectively, shows that

$$\mathbb{P}_0\Big[Z_{N,K}(\delta) \geq 1\Big] \;\; \geq \;\; (1 - d_\kappa^{(\kappa)}(0, \delta/\alpha))\Big(1 - \big(\widehat{d}\,\big)^{\circ J}(0)\Big), \tag{5.2.18}$$

where $(\widehat{d}\,)^{\circ J} := \underbrace{\widehat{d} \circ \cdots \circ \widehat{d}}_{J}$. Going back to the generating function $\widehat{d}(s) = d_a^{(a)}(s, \frac{\delta}{\alpha}) = \mathbb{E}_{\delta/\alpha}[s^{D_a^{(a)}}]$, we see that

$$\mathbb{E}_{\delta/\alpha}[D_a^{(a)}] = \frac{(a\delta/\alpha)^a}{a!} = (e_a\delta/\alpha)^a, \tag{5.2.19}$$

where $e_a := (\frac{a^a}{a!})^{1/a}$. Stirling's approximation shows that

$$2 < \frac{n!}{\sqrt{n}(n/e)^n} < 3, \quad \forall n \geq 1. \tag{5.2.20}$$

It follows that $e_a \uparrow e$ as $a \uparrow \infty$. For $\delta > \alpha/e$, there exists an integer $a(\delta, \alpha)$ such that $e_a \delta/\alpha > 1$ for all $a \geq a(\delta, \alpha)$. This implies that

$$\widehat{d}\,'(1) = \mathbb{E}_{\delta/\alpha}[D_a^{(a)}] > 1, \quad \forall a \geq a(\delta, \alpha). \tag{5.2.21}$$

Thus, for the Galton-Watson tree whose offspring has generating function $\widehat{d}(s)$, its extinction probability, denoted by $\widehat{q}(a, \delta/\alpha)$, satisfies that

$$\widehat{q}(a, \delta/\alpha) = \lim_{J \to \infty} \big(\widehat{d}\,\big)^{\circ J}(0) < 1. \tag{5.2.22}$$

This tells us that

$$\Big(1 - \big(\widehat{d}\,\big)^{\circ J}(0)\Big) \geq 1 - \widehat{q}(a, \delta/\alpha) =: \widehat{p}(a, \delta/\alpha) > 0, \quad \forall J \geq 0. \tag{5.2.23}$$

Moreover, for any $a > 0$ fixed, we have

$$\beta(a, \delta/\alpha) := \inf_{0 \leq \kappa < a} \Big(1 - d_\kappa^{(\kappa)}(0, \delta/\alpha)\Big) > 0, \tag{5.2.24}$$

as $d_\kappa^{(\kappa)}$ are non-trivial generating functions.

Therefore, we end up with

$$\inf_{N \geq a(\delta, \alpha)} \mathbb{P}\Big[Z_{N, \alpha N}(\delta) \geq 1\Big] \geq c_0(\delta, \alpha) > 0, \tag{5.2.25}$$

where $c_0(\delta, \alpha) := \beta(a(\delta, \alpha), \delta/\alpha)\widehat{p}(a(\delta, \alpha), \delta/\alpha) > 0$.

Notice that $\mathbb{P}[Z_{N,\alpha N}(\delta) > 0] > 0$ for any $1 \leq N \leq a(\delta, \alpha)$. We conclude the proof of this lemma by taking $c(\delta, \alpha) := \min_{1 \leq N \leq a(\delta, \alpha)} \{c_0(\delta, \alpha), \mathbb{P}[Z_{N,\alpha N}(\delta) > 0]\} > 0$. $\square$

Now we are ready to prove Theorem 5.1.1.

Proof of Theorem 5.1.1. For $0 < \alpha < e$, let $\delta \in (\frac{\alpha}{e}, 1 \wedge \alpha)$. Observe that under $\mathbb{P}_0$,

$$Z_{N, \alpha N} \geq \sum_{|\omega|=1} 1_{(0 < x_\omega < 1 - \delta)} \sum_{|\sigma| = \alpha N} 1_{(\sigma_1 = \omega)} 1_{(1 - \delta < x_{\sigma_2} < \cdots < x_\sigma \leq 1)}. \tag{5.2.26}$$

For all vertex ω in the first generation, the variables $\sum_{|\sigma|=\alpha N} 1_{(\sigma_1=\omega)} 1_{(1-\delta<x_{\sigma_2}<\cdots<x_\sigma\leq 1)}$ are independent and distributed as $Z_{N,\alpha N-1}(\delta)$. Consequently,

$$\begin{aligned}
\mathbb{P}_0\Big[Z_{N,\alpha N}=0\Big] &\leq \mathbb{P}_0\Big[\sum_{|\omega|=1} 1_{(0<x_\omega<1-\delta)} \sum_{|\sigma|=\alpha N} 1_{(\sigma_1=\omega)} 1_{(1-\delta<x_{\sigma_2}<\cdots<x_\sigma\leq 1)}=0\Big] \\
&= \left(\mathbb{P}_0\Big[1_{(0<x_\omega<1-\delta)} \sum_{|\sigma|=\alpha N} 1_{(\sigma_1=\omega)} 1_{(1-\delta<x_{\sigma_2}<\cdots<x_\sigma\leq 1)}=0\Big]\right)^N \\
&= \left(\delta+(1-\delta)\mathbb{P}_0\Big[Z_{N,\alpha N-1}(\delta)=0\Big]\right)^N.
\end{aligned}$$

By Lemma 5.2.2, $\mathbb{P}_0\Big[Z_{N,\alpha N-1}(\delta)=0\Big] \leq \mathbb{P}_0\Big[Z_{N,\alpha N}(\delta)=0\Big] \leq 1-c(\delta,\alpha)$. Thus,

$$\mathbb{P}_0\Big[Z_{N,\alpha N}=0\Big] \leq \left(\delta+(1-\delta)\Big(1-c(\delta,\alpha)\Big)\right)^N \leq e^{-c(\delta,\alpha)(1-\delta)N}, \tag{5.2.27}$$

which converges to zero as N goes to infinity. This tells us that

$$\lim_{N\to\infty} \mathbb{P}_0\Big[Z_{N,\alpha N}\geq 1\Big]=1, \tag{5.2.28}$$

which is Part (1) of Theorem 5.1.1.

It remains to prove Part (2). Recall that for any $\forall x\in[0,1]$ and $\forall k\geq 1$, $\mathbb{E}_x\Big[Z_{N,k}\Big]=N^k\frac{(1-x)^k}{k!}$. Recall also that $\theta(\alpha)=\alpha(1-\log\alpha)$ for $\alpha>0$. Now take $k=\alpha N$ and $x=0$. Applying Stirling's formula (5.2.20) yields that

$$\mathbb{E}_0\Big[Z_{N,\alpha N}\Big] \leq \frac{(e/\alpha)^{\alpha N}}{2\sqrt{\alpha N}} = \frac{e^{\theta(\alpha)N}}{2\sqrt{\alpha N}}. \tag{5.2.29}$$

For $\alpha\geq e$, note that $\theta(\alpha)\leq 0$. By Markov's inequality,

$$\mathbb{P}_0\Big[Z_{N,\alpha N}\geq 1\Big] \leq \mathbb{E}_0\Big[Z_{N,\alpha N}\Big] \leq \frac{1}{2\sqrt{\alpha N}}, \tag{5.2.30}$$

which converges to zero as N goes to infinity. This completes the proof of Theorem 5.1.1. □

5.3 Proof of Theorem 5.1.3

5.3.1 Typical accessible paths

We begin with two lemmas, which estimate the probabilities of some typical accessible paths.

Lemma 5.3.1. *Let $\{U_j; j\geq 1\}$ be a sequence of i.i.d. $U[0,1]$ random variables.*

1) *For any* $1 \le j \le J-1$,

$$\phi(j,J) := \mathbb{P}\Big(U_1 \le \cdots \le U_j; U_i \ge \frac{i}{J}, \forall 1 \le i \le j\Big) = \frac{J-j}{j!J}. \tag{5.3.1}$$

2) *For any* $\varepsilon \in [0,1)$ *and* $1 \le j \le J$,

$$\begin{aligned}\psi(j,J,\varepsilon) &:= \mathbb{P}\Big(U_1 \le \cdots \le U_j; U_i \ge \varepsilon + (1-\varepsilon)\frac{i-1}{J}, \forall 1 \le i \le j\Big)\\ &= \frac{(1+1/J)^j(J+1-j)}{j!(J+1)}(1-\varepsilon)^j.\end{aligned} \tag{5.3.2}$$

This lemma slightly generalizes Lemma 2 in [129] which gives that $\phi(j,j+1) = \frac{1}{(j+1)!}$.

Proof. According to the assumption, we compute $\phi(j,J)$ directly.

$$\begin{aligned}\phi(j,J) &= \int_{j/J}^{1}\int_{(j-1)/J}^{u_j}\cdots\int_{1/J}^{u_2} du_1\cdots du_j\\ &= \int_{j/J}^{1}\int_{(j-1)/J}^{u_j}\cdots\int_{i/J}^{u_{i+1}}\Big(\frac{u_i^{i-1}}{(i-1)!} - \frac{1}{J}\frac{u_i^{i-2}}{(i-2)!}\Big)du_i\cdots du_j\\ &= \frac{J-j}{j!J},\end{aligned}$$

giving (5.3.1). We now compute ψ by using ϕ. Notice that when $\varepsilon = \frac{1}{J+1}$, $\varepsilon + (1-\varepsilon)\frac{i-1}{J} = \frac{i}{J+1}$ for any $1 \le i \le j$. Hence,

$$\psi(j,J,\frac{1}{J+1}) = \phi(j,J+1) = \frac{J+1-j}{j!(J+1)}. \tag{5.3.3}$$

On the other hand, we rewrite $\psi(j,J,\varepsilon)$ as follows :

$$\psi(j,J,\varepsilon) = \int_{\varepsilon+(1-\varepsilon)\frac{j-1}{J}}^{1}\cdots\int_{\varepsilon}^{u_2} du_1\cdots du_j.$$

Take $u_i = \varepsilon + (1-\varepsilon)v_i$ for all $1 \le i \le j$. By a change of variables,

$$\psi(j,J,\varepsilon) = \int_{(j-1)/J}^{1}\cdots\int_{0}^{v_2}(1-\varepsilon)^j dv_1\cdots dv_j = (1-\varepsilon)^j\psi(j,J,0).$$

By (5.3.3), we get that $\psi(j,J,0) = \psi(j,J,\frac{1}{J+1})(1-\frac{1}{J+1})^{-j} = \frac{(1+1/J)^j(J+1-j)}{j!(J+1)}$. Therefore,

$$\psi(j,J,\varepsilon) = (1-\varepsilon)^j\psi(j,J,0) = \frac{(1+1/J)^j(J+1-j)}{j!(J+1)}(1-\varepsilon)^j, \tag{5.3.4}$$

as desired. □

Following the assumption of Lemma 5.3.1, we define for any $0 \le L < K$,

$$A_L(K) := \{U_1 < \cdots < U_K; U_j \ge \frac{(j-L)_+}{K+1}; \forall 1 \le j \le K\}. \tag{5.3.5}$$

Obviously, $\mathbb{P}[A_0(K)] = \phi(K, K+1) = \frac{1}{(K+1)!}$ by (5.3.1).

Lemma 5.3.2. *There exists a positive constant $c_0 > 0$ such that for any $1 \le L < K$,*

$$\mathbb{P}\Big[A_L(K)\Big] \le \frac{e^{c_0\sqrt{L}}}{K^{3/2}} \frac{e^K}{(K+1)^K}. \tag{5.3.6}$$

Proof. Clearly, $\mathbb{P}[A_1(K)] = \psi(K, K, 0) = \frac{(1+1/K)^K}{(K+1)!}$ by (5.3.2). By (5.2.20),

$$\mathbb{P}\Big[A_L(K)\Big] \le \frac{e^{2\sqrt{L}}}{K^{3/2}} \frac{e^K}{(K+1)^K}, \text{ for } L = 1. \tag{5.3.7}$$

The fact $A_1(K) \subset A_2(K) \subset \cdots \subset A_L(K)$ leads to

$$\mathbb{P}[A_L(K)] = \sum_{i=1}^{L-1} \mathbb{P}[A_{i+1}(K) \setminus A_i(K)] + \mathbb{P}[A_1(K)], \quad 2 \le L < K. \tag{5.3.8}$$

Let us estimate $\mathbb{P}[A_{i+1}(K) \setminus A_i(K)]$. Observe that

$$\mathbb{P}[A_{i+1}(K) \setminus A_i(K)] = \sum_{k=i+1}^{K} \mathbb{P}[C_{i,k}(K)], \tag{5.3.9}$$

where

$$C_{i,k}(K) := \left\{ \begin{array}{rl} U_1 < \cdots < U_K; U_j & \ge \frac{j-i}{K+1}, \forall i+1 \le j \le k-1; \\ & U_k < \frac{k-i}{K+1}; U_j \ge \frac{j-i-1}{K+1}, \forall k+1 \le j \le K \end{array} \right\}. \tag{5.3.10}$$

It follows from the independence of U_j's that $\mathbb{P}[C_{i,k}(K)] = p_{i,k} q_{i,k}$ where

$$\begin{aligned} p_{i,k} &:= \mathbb{P}\Big(U_1 < \cdots < U_k < \frac{k-i}{K+1}; U_j \ge \frac{j-i}{K+1}, \forall i+1 \le j \le k-1\Big); \\ q_{i,k} &:= \mathbb{P}\Big(\frac{k-i}{K+1} \le U_{k+1} < \cdots < U_K; U_j \ge \frac{j-i-1}{K+1}, \forall k+1 \le j \le K\Big). \end{aligned}$$

Then (5.3.9) becomes that

$$\mathbb{P}[A_{i+1}(K) \setminus A_i(K)] = \sum_{k=i+1}^{K} p_{i,k} q_{i,k}. \tag{5.3.11}$$

We first compute $q_{i,k}$:

$$\begin{aligned} q_{i,k} &= \mathbb{P}\Big(U_1 < \cdots < U_{K-k}; U_j \geq \frac{j+k-i-1}{K+1}, \forall 1 \leq j \leq K-k\Big) \\ &= \psi(K-k, K-k+i+1, \frac{k-i}{K+1}). \end{aligned}$$

By (5.3.2), we obtain that

$$q_{i,k} = \Big(\frac{K+2+i-k}{K+1}\Big)^{K-k} \frac{i+2}{(K-k)!(K-k+i+2)}. \tag{5.3.12}$$

It remains to estimate $p_{i,k}$. One sees that

$$\begin{aligned} p_{i,k} &= \Big(\frac{k-i}{K+1}\Big)^k \mathbb{P}\Big(U_1 < \cdots < U_k; U_j \geq \frac{j-i}{k-i}, \forall i+1 \leq j \leq k-1\Big) \\ &\leq \Big(\frac{k-i}{K+1}\Big)^k \frac{1}{k-i} \mathbb{P}(D_{i,k-1}), \end{aligned} \tag{5.3.13}$$

where

$$D_{i,k} := \Big\{U_1 < \cdots < U_k, U_j \geq \frac{j-i}{k-i+1}, \forall i+1 \leq j \leq k\Big\},\ k \geq i \geq 1. \tag{5.3.14}$$

Let us admit for the moment the following lemma, whose proof will be given later.

Lemma 5.3.3. *For $k \geq i \geq 1$, there exists a constant $c_1 > 0$ such that*

$$u_{i,k} := \mathbb{P}\Big(D_{i,k}\Big) \leq \frac{e^{k-i} e^{c_1\sqrt{i-1}+2}}{(k+1-i)^k k^{3/2}}. \tag{5.3.15}$$

Lemma 5.3.3 implies that

$$\begin{aligned} p_{i,k} &\leq \Big(\frac{k-i}{K+1}\Big)^k \frac{1}{k-i} u_{i,k-1} \\ &\leq \Big(\frac{e}{K+1}\Big)^k \frac{e^{-i-1} e^{c_1\sqrt{i-1}+2}}{(k-1)^{3/2}}. \end{aligned} \tag{5.3.16}$$

Let us go back to (5.3.11). In view of (5.3.12) and (5.3.16), we see that

$$\begin{aligned} &\mathbb{P}[A_{i+1}(K) \setminus A_i(K)] = \sum_{k=i+1}^{K} p_{i,k} q_{i,k} \\ &\quad \leq \sum_{k=i+1}^{K} \Big(\frac{K+2+i-k}{K+1}\Big)^{K-k} \frac{i+2}{(K-k)!(K-k+i+2)} \Big(\frac{e}{K+1}\Big)^k \frac{e^{-i-1} e^{c_1\sqrt{i-1}+2}}{(k-1)^{3/2}}. \end{aligned} \tag{5.3.17}$$

Applying Stirling's formula (5.2.20) to $(K-k)!$ yields that

$$\begin{aligned}\mathbb{P}[A_{i+1}(K)\setminus A_i(K)] &\leq \frac{(i+2)e^{c_1\sqrt{i-1}+2}e^K}{(K+1)^K}\sum_{k=i}^{K-1}\frac{e}{2k^{3/2}(K+i+1-k)^{3/2}}\\ &\leq c_2\frac{\sqrt{i}e^{c_1\sqrt{i-1}+2}}{(K+1)^{3/2}}\frac{e^K}{(K+1)^K}.\end{aligned}$$

We then deduce from (5.3.8) that for $L\geq 2$,

$$\mathbb{P}[A_L(K)]=\sum_{i=1}^{L-1}\mathbb{P}[A_{i+1}(K)\setminus A_i(K)]+\mathbb{P}[A_1(K)]\leq c_3\frac{L^{3/2}e^{c_1\sqrt{L-1}+2}}{K^{3/2}}\frac{e^K}{(K+1)^K}, \tag{5.3.18}$$

which is sufficient to conclude Lemma 5.3.2. □

We now present the proof of Lemma 5.3.3.

Proof of Lemma 5.3.3. Recall that $D_{i,k}=\Big\{U_1<\cdots<U_k, U_j\geq\frac{j-i}{k-i+1},\forall i+1\leq j\leq k\Big\}$. Since $D_{i,k}\subset\{U_1<\cdots<U_k\}$, we have for any $k\geq i$,

$$u_{i,k}=\mathbb{P}\Big(D_{i,k}\Big)\leq\mathbb{P}\Big(U_1<\cdots<U_k\Big)=\frac{1}{k!}. \tag{5.3.19}$$

By Stirling's formula (5.2.20), we get that

$$u_{i,k}\leq\frac{e^k}{2k^k\sqrt{k}}=\frac{e^k}{(k+1-i)^kk^{3/2}}\Big(1-\frac{i-1}{k}\Big)^k\frac{k}{2}\leq\frac{e^{k+1-i}}{(k+1-i)^kk^{3/2}}\frac{k}{2},$$

as $1-z\leq e^{-z}$ for any $z\geq 0$. Take $c_1:=\max\{40,\sup_{i\geq 2}\frac{1+\log i}{\sqrt{i-1}}\}<\infty$. Then when $k\leq 2i$, we deduce that

$$u_{i,k}\leq\frac{e^{k-i}}{(k+1-i)^kk^{3/2}}e^{\log i+1}\leq\frac{e^{k-i}e^{c_1\sqrt{i-1}+2}}{(k+1-i)^kk^{3/2}}. \tag{5.3.20}$$

It remain to prove the inequality (5.3.15) when $k/2\geq i\geq 1$. Let $\gamma(i):=e^{c_1\sqrt{i-1}+2}$. According to Lemma 5.3.1, we have

$$u_{1,k}=\psi(k,k,0)=\frac{(1+\frac{1}{k})^k}{(k+1)!}\leq\frac{e^{k+1}}{2k^{k+3/2}}\leq\frac{e^{k-1}\gamma(1)}{(k+1-1)^kk^{3/2}},\ \forall k\geq 1, \tag{5.3.21}$$

giving (5.3.15) in case $i=1$.

We prove (5.3.15) by induction on i. Assume (5.3.15) for some $i\geq 1$ (and all $k\geq i$). We need to bound $\mathbb{P}(D_{i+1,k})$ for $k\geq 2(i+1)$.

Since $D_{1,k} \subset D_{2,k} \subset \cdots D_{k-1,k}$, we have :

$$\begin{aligned} u_{i+1,k} - u_{i,k} &= \mathbb{P}\Big(D_{i+1,k} \setminus D_{i,k}\Big) \\ &= \sum_{j=1}^{k-i} \mathbb{P}\Big(U_1 < \cdots < U_k, U_{i+\ell} \geq \frac{1}{k-i+1}, \forall 1 \leq \ell < j; \frac{j-1}{k-i} \leq U_{i+j} < \frac{j}{k-i+1}; \\ &\qquad \frac{j}{k-i+1} < \frac{j+\ell-1}{k-i} \leq U_{i+j+\ell}, \forall 1 \leq \ell \leq k-j-i\Big). \end{aligned} \tag{5.3.22}$$

By the independence of the U_i's, we have $u_{i+1,k} - u_{i,k} = \sum_{j=1}^{k-i} r_{i,j,k} s_{i,j,k}$ where

$$\begin{aligned} r_{i,j,k} &:= \mathbb{P}\Big(U_1 < \cdots < U_{i+j}, U_{i+\ell} \geq \frac{\ell}{k-i+1}, \forall 1 \leq \ell < j; \frac{j-1}{k-i} \leq U_{i+j} < \frac{j}{k-i+1}\Big) \\ s_{i,j,k} &:= \mathbb{P}\Big(U_{i+j+1} < \cdots < U_k; \frac{j+\ell-1}{k-i} \leq U_{i+j+\ell}, \forall 1 \leq \ell \leq k-j-i\Big). \end{aligned}$$

Once again by (5.3.2),

$$s_{i,j,k} = \psi(k-i-j, k-i-j, \frac{j}{k-i}) = \left(\frac{k-i-j+1}{k-i}\right)^{k-i-j} \frac{1}{(k-i-j+1)!}. \tag{5.3.23}$$

On the other hand,

$$\begin{aligned} r_{i,j,k} &\leq \mathbb{P}\Big(U_1 \leq \cdots \leq U_{i+j-1} \leq \frac{j}{k-i+1}, U_{i+\ell} \geq \frac{\ell}{k-i+1}, \forall 1 \leq \ell < j\Big) \times \left[\frac{j}{k-i+1} - \frac{j-1}{k-i}\right] \\ &= \left(\frac{j}{k-i+1}\right)^{i+j-1} \mathbb{P}\Big(U_1 \leq \cdots \leq U_{i+j-1}, U_{i+\ell} \geq \frac{\ell}{j}, \forall 1 \leq \ell \leq j-1\Big) \frac{k-i-j+1}{(k-i)(k-i+1)} \\ &= \left(\frac{j}{k-i+1}\right)^{i+j-1} \frac{k-i-j+1}{(k-i)(k-i+1)} u_{i,i+j-1}. \end{aligned}$$

This implies that

$$\begin{aligned} u_{i+1,k} - u_{i,k} &= \sum_{j=1}^{k-i} r_{i,j,k} s_{i,j,k} \\ &\leq \sum_{j=1}^{k-i} \left(\frac{k-i-j+1}{k-i}\right)^{k-i-j} \frac{1}{(k-i-j+1)!} \left(\frac{j}{k-i+1}\right)^{i+j-1} \frac{k-i-j+1}{(k-i)(k-i+1)} u_{i,i+j-1}. \end{aligned} \tag{5.3.24}$$

By induction assumption, for any $\ell \geq i \geq 1$, $u_{i,\ell} \leq \frac{e^{\ell-i}\gamma(i)}{(\ell+1-i)^{\ell}\ell^{3/2}}$. It follows that

$$\begin{aligned} u_{i+1,k} \leq \frac{e^{k-i}\gamma(i)}{(k+1-i)^k k^{3/2}} &+ \sum_{j=1}^{k-i} \left(\frac{j}{k-i+1}\right)^{i+j-1} \frac{k-i-j+1}{(k-i)(k-i+1)} \\ &\times \frac{e^{j-1}\gamma(i)}{j^{i+j-1}(i+j-1)^{3/2}} \left(\frac{k-i-j+1}{k-i}\right)^{k-i-j} \frac{1}{(k-i-j+1)!}. \end{aligned}$$

The first term on the right-hand side of this inequality is bounded by

$$\frac{e^{k-i}\gamma(i)}{(k-i)^k k^{3/2}}\Big(\frac{k-i}{k+1-i}\Big)^{k+1-i} \leq \frac{e^{k-i-1}\gamma(i)}{(k-i)^k k^{3/2}}, \tag{5.3.25}$$

whereas the second term bounded by

$$\begin{aligned} & \sum_{j=1}^{k-i}\Big(\frac{1}{k-i}\Big)^{k+1}\frac{e^{j-1}\gamma(i)}{(i+j-1)^{3/2}}\Big(k-i-j+1\Big)^{k-i-j+1}\frac{1}{(k-i-j+1)!} \\ \leq\ & \frac{e^{k-i}\gamma(i)}{(k-i)^{k+1}}\sum_{j=1}^{k-i}\frac{1}{2(i+j-1)^{3/2}(k-i-j+1)^{1/2}} \\ \leq\ & \frac{20\gamma(i)}{\sqrt{i}}\frac{e^{k-i-1}}{(k-i)^k k^{3/2}}, \end{aligned}$$

where the last inequality holds as we take $k/2 \geq i+1$. We obtain that

$$u_{i+1,k} \leq \frac{e^{k-i-1}\gamma(i)}{(k-i)^k k^{3/2}}\Big(1+\frac{20}{\sqrt{i}}\Big) \leq \frac{e^{k-i-1}}{(k-i)^k k^{3/2}}\gamma(i)e^{\frac{20}{\sqrt{i}}}. \tag{5.3.26}$$

Note that $\gamma(i)e^{\frac{20}{\sqrt{i}}} = \exp\{c_1\sqrt{i-1}+2+\frac{20}{\sqrt{i}}\} \leq \gamma(i+1)$ if we take $c_1 > 40$. Therefore,

$$u_{i+1,k} \leq \frac{e^{k-i-1}\gamma(i+1)}{(k-i)^k k^{3/2}},\ \forall k \geq i+1, \tag{5.3.27}$$

which completes the proof of Lemma 5.3.3. □

5.3.2 Proof of Theorem 5.1.3

We prove for the three cases separately.

Proof of (i) of Theorem 5.1.3. We need to show that for $\alpha \in (0,e)$, $\mathbb{P}_0$-almost surely,

$$\lim_{N\to\infty} \log Z_{N,\alpha N}/N = \theta(1-\alpha)$$

with $\theta(\alpha) = \alpha(1-\log\alpha)$. First of all, let us prove the upper bound.

Recall that

$$\mathbb{E}_0\Big[Z_{N,\alpha N}\Big] \leq \frac{(e/\alpha)^{\alpha N}}{2\sqrt{\alpha N}} = \frac{e^{\theta(\alpha)N}}{2\sqrt{\alpha N}}. \tag{5.3.28}$$

By Markov's inequality, for any $\delta > 0$,

$$\mathbb{P}_0\Big[Z_{N,\alpha N} \geq \exp\{N(\theta(\alpha)+\delta)\}\Big] \leq \exp\{-N(\theta(\alpha)+\delta)\}\mathbb{E}_0\Big[Z_{N,\alpha N}\Big] \leq \frac{e^{-\delta N}}{2\sqrt{\alpha N}}, \tag{5.3.29}$$

which is summable in N. By the Borel-Cantelli lemma, we have $\mathbb{P}_0$-almost surely,

$$\limsup_{N\to\infty} \frac{\log Z_{N,\alpha N}}{N} \le \theta(\alpha). \tag{5.3.30}$$

It remains to prove the lower bound. For any $\varepsilon \in (0,1)$ and any $k \ge 1$, let $\mathscr{A}_{N,k,\varepsilon} := \{\sigma \in \mathscr{A}_{N,k}; x_{\sigma_i} \ge \varepsilon + (1-\varepsilon)\frac{i-1}{k}, \forall 1 \le i \le k\}$. We define the following quantities :

$$Z_{N,k,\varepsilon} := \sum_{|\sigma|=k} 1_{(\sigma\in\mathscr{A}_{N,k,\varepsilon})}, \quad \forall k \ge 1. \tag{5.3.31}$$

Clearly, $Z_{N,k,\varepsilon} \le Z_{N,k}$. Instead of $Z_{N,k}$, we study $Z_{N,k,\varepsilon}$.

By (5.3.2), we see that for any $k \ge 1$ and any $\varepsilon \in (0,1)$,

$$\mathbb{E}_\varepsilon\Big[Z_{N,k,\varepsilon}\Big] = N^k \psi(k,k,\varepsilon) = N^k \frac{(1+1/k)^k}{(k+1)!}(1-\varepsilon)^k. \tag{5.3.32}$$

Here we take $k = \alpha N - 1$ with $\alpha < e$. For any α fixed, let ε be small enough so that $\alpha < e(1-\varepsilon)$ and $\log(1-\varepsilon) > -2\varepsilon$. By Stirling's formula (5.2.20),

$$\mathbb{E}_\varepsilon\Big[Z_{N,\alpha N-1,\varepsilon}\Big] \ge c_4 \frac{\exp\Big\{\theta(\alpha)N + \alpha\log(1-\varepsilon)N\Big\}}{(\alpha N)^{3/2}}. \tag{5.3.33}$$

For ε sufficiently small such that $\theta(\alpha) > 3\alpha\varepsilon$ and N sufficiently large, we get that

$$\mathbb{E}_\varepsilon\Big[Z_{N,\alpha N-1,\varepsilon}\Big] \ge 2\exp\{\theta(\alpha)N - 3\alpha\varepsilon N\} \ge 1. \tag{5.3.34}$$

By the Paley-Zygmund inequality,

$$\mathbb{P}_\varepsilon\Big[Z_{N,\alpha N-1,\varepsilon} \ge \exp\{\theta(\alpha)N - 3\alpha\varepsilon N\}\Big] \ge \frac{\mathbb{E}_\varepsilon\Big[Z_{N,\alpha N,\varepsilon}\Big]^2}{4\mathbb{E}_\varepsilon\Big[Z^2_{N,\alpha N,\varepsilon}\Big]}. \tag{5.3.35}$$

Let us bound $\mathbb{E}_\varepsilon\Big[Z^2_{N,\alpha N,\varepsilon}\Big]$, which can be written as follows :

$$\begin{aligned}
\mathbb{E}_\varepsilon\Big[Z^2_{N,k,\varepsilon}\Big] &= \mathbb{E}_\varepsilon\Big[Z_{N,k,\varepsilon}\Big] + \mathbb{E}_\varepsilon\Big[\sum_{q=0}^{k-1}\sum_{|\sigma\wedge\sigma'|=q} 1_{(\sigma,\sigma'\in\mathscr{A}_{N,k,\varepsilon})}\Big] \qquad (5.3.36)\\
&= \mathbb{E}_\varepsilon\Big[Z_{N,k,\varepsilon}\Big] + \sum_{q=0}^{k-1} N^q N(N-1)N^{2k-2q-2}\mathbb{P}_\varepsilon\Big(\sigma,\sigma' \in \mathscr{A}_{N,k,\varepsilon}\Big| |\sigma\wedge\sigma'| = q\Big),
\end{aligned}$$

where $\sigma \wedge \sigma'$ denotes the latest common ancestor of σ and σ'.

Recall that $\mathscr{A}_{N,k,\varepsilon} = \{\sigma \in \mathscr{A}_{N,k}; x_{\sigma_i} \geq \varepsilon + (1-\varepsilon)\frac{i-1}{k}, \forall 1 \leq i \leq k\}$. We have

$$\begin{aligned}
& \mathbb{P}_\varepsilon\Big(\sigma,\sigma' \in \mathscr{A}_{N,k,\varepsilon}\Big| |\sigma \wedge \sigma'| = q\Big) \\
= & \int_{\varepsilon+(1-\varepsilon)(q-1)/k}^{1} \mathbb{P}_\varepsilon\Big(\sigma,\sigma' \in \mathscr{A}_{N,k,\varepsilon}\Big| |\sigma \wedge \sigma'| = q, x_{\sigma_q} = y\Big) dy \\
= & \int_{\varepsilon+(1-\varepsilon)(q-1)/k}^{1} \mathbb{P}\Big(U_1 < \cdots < U_{q-1} < y; U_i \geq \varepsilon + (1-\varepsilon)\frac{i-1}{k}, \forall 1 \leq i < q\Big) \\
& \times \Big[\mathbb{P}\Big(y < U_{q+1} < \cdots < U_k; U_i \geq \varepsilon + (1-\varepsilon)\frac{i-1}{k}, \forall q < i \leq k\Big)\Big]^2 dy. \qquad (5.3.37)
\end{aligned}$$

Observe that

$$\mathbb{P}\Big(y < U_{q+1} < \cdots < U_k; U_i \geq \varepsilon + (1-\varepsilon)\frac{i-1}{k}, \forall q < i \leq k\Big) \leq \psi(k-q, k-q, \varepsilon + (1-\varepsilon)\frac{q}{k}).$$

This implies that

$$\mathbb{P}_\varepsilon\Big(\sigma,\sigma' \in \mathscr{A}_{N,k,\varepsilon}\Big| |\sigma \wedge \sigma'| = q\Big) \leq \psi(k-q, k-q, \varepsilon + (1-\varepsilon)\frac{q}{k}) \times \psi(k,k,\varepsilon). \qquad (5.3.38)$$

Combining (5.3.36) with (5.3.38) yields that

$$\begin{aligned}
& \mathbb{E}_\varepsilon\Big[Z^2_{N,k,\varepsilon}\Big] \\
\leq & \mathbb{E}_\varepsilon\Big[Z_{N,k,\varepsilon}\Big] + \frac{N-1}{N} \sum_{q=0}^{k-1} N^{2k-q} \psi(k-q, k-q, \varepsilon + (1-\varepsilon)\frac{q}{k}) \times \psi(k,k,\varepsilon) \\
= & \mathbb{E}_\varepsilon\Big[Z_{N,k,\varepsilon}\Big]\Big(1 + \frac{N-1}{N}\mathbb{E}_\varepsilon\Big[Z_{N,k,\varepsilon}\Big] \sum_{q=0}^{k-1} N^{-q} \frac{\psi(k-q, k-q, \varepsilon + (1-\varepsilon)\frac{q}{k})}{\psi(k,\varepsilon)}\Big), \qquad (5.3.39)
\end{aligned}$$

where the last equality follows from (5.3.32). By (5.3.2) and (5.2.20),

$$\sum_{q=0}^{k-1} N^{-q} \frac{\psi(k-q, k-q, \varepsilon + (1-\varepsilon)\frac{q}{k})}{\psi(k,\varepsilon)} \leq \sum_{q=0}^{k-1} c_5 \Big(\frac{k}{k-q}\Big)^{3/2} \Big(\frac{k}{e(1-\varepsilon)N}\Big)^q. \qquad (5.3.40)$$

For $k = \alpha N - 1$ and $\alpha < e(1-\varepsilon)$, we get that for N large enough,

$$\sum_{q=0}^{k-1} c_5 \Big(\frac{k}{k-q}\Big)^{3/2} \Big(\frac{k}{e(1-\varepsilon)N}\Big)^q \leq \sum_{q=0}^{k/2} c_6 \Big(\frac{\alpha}{e(1-\varepsilon)}\Big)^q + \sum_{q \geq k/2} c_6 q^{3/2} \Big(\frac{\alpha}{e(1-\varepsilon)}\Big)^q \leq c_7 < \infty.$$

Recall that for ε small enough and N large enough, $\mathbb{E}_\varepsilon\Big[Z_{N,k,\varepsilon}\Big] \geq 1$. Going back to (5.3.39), we obtain that for $\alpha < e(1-\varepsilon)$ with ε close to zero and for all N sufficiently large,

$$\mathbb{E}_\varepsilon\Big[Z^2_{N,\alpha N-1,\varepsilon}\Big] \leq (1+c_7)\mathbb{E}_\varepsilon\Big[Z_{N,\alpha N-1,\varepsilon}\Big]^2. \qquad (5.3.41)$$

It then follows from (5.3.35) that

$$\mathbb{P}_\varepsilon\Big[Z_{N,\alpha N-1,\varepsilon} \geq \exp\{\theta(\alpha)N - 3\alpha\varepsilon N\}\Big] \geq \frac{1}{4(1+c_7)} =: c_8 \in (0,1). \tag{5.3.42}$$

For any vertex ω in the first generation, define $\mathscr{A}_{N,k,\varepsilon}(\omega)$ as follows :

$$\mathscr{A}_{N,k,\varepsilon}(\omega) := \{|\sigma| = k+1; \sigma_1 = \omega; x_{\sigma_2} < \cdots < x_\sigma; x_{\sigma_i} \geq \varepsilon + (1-\varepsilon)\frac{i-2}{k}, 2 \leq i \leq k+1\}.$$

To bound $\mathbb{P}_0\{Z_{N,\alpha N} < \exp\{\theta(\alpha)N - 3\alpha\varepsilon N\}\}$, we observe that

$$Z_{N,\alpha N} \geq \sum_{|\omega|=1} 1_{(x_\omega<\varepsilon)} \sum_{|\sigma|=\alpha N} 1_{(\sigma\in\mathscr{A}_{N,\alpha N-1,\varepsilon}(\omega))}, \tag{5.3.43}$$

where $\Big(x_\omega, \sum_{|\sigma|=\alpha N} 1_{(\sigma\in\mathscr{A}_{N,\alpha N-1,\varepsilon}(\omega))}\Big)$ are i.i.d. Thus,

$$\begin{aligned}
& \mathbb{P}_0\Big(Z_{N,\alpha N} < \exp\{\theta(\alpha)N - 3\alpha\varepsilon N\}\Big) \\
\leq \;& \mathbb{P}_0\Big(\sum_{|\omega|=1} 1_{(x_\omega<\varepsilon)} \sum_{|\sigma|=\alpha N} 1_{(\sigma\in\mathscr{A}_{N,\alpha N-1,\varepsilon}(\omega))} < \exp\{\theta(\alpha)N - 3\alpha\varepsilon N\}\Big) \\
\leq \;& \mathbb{P}_0\Big(1_{(x_\omega<\varepsilon)} \sum_{|\sigma|=\alpha N} 1_{(\sigma\in\mathscr{A}_{N,\alpha N-1,\varepsilon}(\omega))} < \exp\{\theta(\alpha)N - 3\alpha\varepsilon N\}\Big)^N.
\end{aligned} \tag{5.3.44}$$

The fact that $\mathbb{P}_0[\sigma \in \mathscr{A}_{N,k,\varepsilon}(\omega)|x_\omega < \varepsilon] = \mathbb{P}_\varepsilon[\sigma \in \mathscr{A}_{N,k,\varepsilon}]$ implies that given $\{x_\omega < \varepsilon\}$, $\sum_{|\sigma|=\alpha N} 1_{(\sigma\in\mathscr{A}_{N,\alpha N-1,\varepsilon}(\omega))}$ is distributed as $Z_{N,\alpha N-1,\varepsilon}$ under $\mathbb{P}_\varepsilon$. Therefore, we have

$$\begin{aligned}
& \mathbb{P}_0\Big(1_{(x_\omega<\varepsilon)} \sum_{|\sigma|=\alpha N} 1_{(\sigma\in\mathscr{A}_{N,\alpha N-1,\varepsilon}(\omega))} < \exp\{\theta(\alpha)N - 3\alpha\varepsilon N\}\Big) \\
\leq \;& 1-\varepsilon+\varepsilon\mathbb{P}_0\Big(\sum_{|\sigma|=\alpha N} 1_{(\sigma\in\mathscr{A}_{N,\alpha N-1,\varepsilon}(\omega))} < \exp\{\theta(\alpha)N - 3\alpha\varepsilon N\}\Big| x_\omega < \varepsilon\Big) \\
= \;& 1-\varepsilon+\varepsilon\Big(1-\mathbb{P}_\varepsilon\Big[Z_{N,\alpha N-1,\varepsilon} \geq \exp\{\theta(\alpha)N - 3\alpha\varepsilon N\}\Big]\Big),
\end{aligned}$$

which is bounded by $1-\varepsilon+\varepsilon(1-c_8)$ because of (5.3.42). Plugging this inequality into (5.3.44) yields that

$$\mathbb{P}_0\Big[Z_{N,\alpha N} < \exp\{\theta(\alpha)N - 3\alpha\varepsilon N\}\Big] \leq \Big(1-\varepsilon+\varepsilon(1-c_8)\Big)^N \leq e^{-Nc_8\varepsilon},$$

which is summable in N. By the Borel-Cantelli lemma, we conclude that for ε sufficiently small, $\mathbb{P}_0-$almost surely,

$$\liminf_{N\to\infty} \frac{\log Z_{N,\alpha N}}{N} \geq \theta(\alpha) - 3\alpha\varepsilon, \tag{5.3.45}$$

completing the proof of (i) of Theorem 5.1.3.

Before the proof of Part (ii), we turn to estimate $\mathbb{P}_0\Big[Z_{N,\alpha N}\geq 1\Big]$ with $\alpha > e$.

Proof of (iii) of Theorem 5.1.3. The upper bound is easy. By Markov's inequality and (5.2.29), one sees that

$$\mathbb{P}_0\Big[Z_{N,\alpha N}\geq 1\Big]\leq \mathbb{E}_0\Big[Z_{N,\alpha N}\Big]\leq \frac{e^{\theta(\alpha)N}}{2\sqrt{\alpha N}}.$$

It follows that

$$\limsup_{N\to\infty}\frac{\log\mathbb{P}_0\Big[Z_{N,\alpha N}\geq 1\Big]}{N}\leq \theta(\alpha)<0. \tag{5.3.46}$$

To get the lower bound, we use the fact that $Z_{N,k}\geq Z_{N,k,\varepsilon}$ to get that for any $x\in[0,1]$ and any $\varepsilon\in(0,1)$,

$$\mathbb{P}_x\Big[Z_{N,k}\geq 1\Big]\geq \mathbb{P}_x\Big[Z_{N,k,\varepsilon}\geq 1\Big]\geq \frac{\mathbb{E}_x\Big[Z_{N,k,\varepsilon}\Big]^2}{\mathbb{E}_x\Big[Z^2_{N,k,\varepsilon}\Big]}, \tag{5.3.47}$$

where the second inequality follows from the Cauchy-Schwartz inequality. In this part of proof, we take $x=0$.

Applying (5.3.32) and Stirling's formula (5.2.20) gives that for $k=\alpha N$,

$$\mathbb{E}_0\Big[Z_{N,k,0}\Big]\geq \frac{N^k}{(k+1)!}\geq \frac{e^{\theta(\alpha)N}}{3(\alpha N+1)^{3/2}}. \tag{5.3.48}$$

On the other hand, in view of (5.3.39), we obtain that

$$\mathbb{E}_0\Big[Z^2_{N,k,0}\Big]\leq \mathbb{E}_0\Big[Z_{N,k,0}\Big]\Big(1+\frac{N-1}{N}\sum_{q=0}^{k-1}N^{k-q}\psi(k-q,k-q,\frac{q}{k})\Big). \tag{5.3.49}$$

By (5.3.2) and (5.2.20), one sees that for $k=\alpha N$ with $\alpha>e$,

$$\begin{aligned}\sum_{q=0}^{k-1}N^{k-q}\psi(k-q,k-q,\frac{q}{k}) &= \sum_{q=0}^{k-1}N^{k-q}\frac{(1+1/(k-q))^{k-q}}{(k-q+1)!}(1-q/k)^{k-q}\\ &\leq \sum_{q=0}^{k-1}\frac{e}{2(k-q)^{3/2}}\Big(\frac{eN}{k}\Big)^{k-q}\leq\sum_{q=0}^{k-1}\frac{e}{2(k-q)^{3/2}},\end{aligned}$$

Let $c_9:=\sum_{q=1}^{\infty}\frac{e}{2q^{3/2}}\in(0,\infty)$. It follows that

$$\sum_{q=0}^{k-1}N^{k-q}\psi(k-q,k-q,\frac{q}{k})\leq c_9. \tag{5.3.50}$$

Plugging it into (5.3.49) shows that

$$\mathbb{E}_0\Big[Z^2_{N,\alpha N,0}\Big] \le \mathbb{E}_0\Big[Z_{N,\alpha N,0}\Big](1+c_9). \tag{5.3.51}$$

According to (5.3.47) and (5.3.48), we obtain that

$$\mathbb{P}_0\Big[Z_{N,\alpha N} \ge 1\Big] \ge \frac{\mathbb{E}_0\Big[Z_{N,\alpha N,0}\Big]}{1+c_9} \ge c_{10}\frac{e^{\theta(\alpha)N}}{(\alpha N+1)^{3/2}}, \tag{5.3.52}$$

where $c_{10} := \frac{1}{3(1+c_9)}$. Therefore, we conclude that for $\alpha > e$,

$$\liminf_{N\to\infty} \frac{\log \mathbb{P}_0\Big[Z_{N,\alpha N} \ge 1\Big]}{N} \ge \theta(\alpha), \tag{5.3.53}$$

which completes the proof of (iii) of Theorem 5.1.3.

Proof of (ii) of Theorem 5.1.3. Let us estimate $\mathbb{P}_0[Z_{N,eN} \ge 1]$.

For the lower bound, one observes that the inequality (5.3.52) still holds when $\alpha = e$. As $\theta(e) = 0$, we get that

$$\mathbb{P}_0[Z_{N,eN} \ge 1] \ge c_{11} N^{-3/2}. \tag{5.3.54}$$

To obtain the upper bound, we introduce the following collections of accessible vertices in $T^{(N)}$:

$$\mathscr{A}_L(K) := \{|\sigma| = K : x_{\sigma_1} < \cdots < x_\sigma; x_{\sigma_j} \ge \frac{(j-L)_+}{K+1}; \forall 1 \le j \le K\}, \quad 0 \le L < K. \tag{5.3.55}$$

Set $K = eN$ and $L_0 = 2\log K$. One observes that

$$\mathscr{A}_{N,K} \subset \mathscr{A}_{L_0}(K) \cup \bigcup_{k=L_0+1}^{K} \{\exists |\sigma| = k : x_{\sigma_1} < \cdots < x_\sigma, x_\sigma < \frac{k-L_0}{K+1}\}. \tag{5.3.56}$$

As a consequence,

$$\begin{aligned}\mathbb{P}_0\Big[Z_{N,eN} \ge 1\Big] &\le \mathbb{P}_0\Big[\exists \sigma \in \mathscr{A}_{L_0}(K)\Big] + \sum_{k=L_0+1}^{K} \mathbb{P}_0\Big[\exists |\sigma| = k : x_{\sigma_1} < \cdots < x_\sigma, x_\sigma < \frac{k-L_0}{K+1}\Big] \\ &\le \mathbb{E}_0\Big[\sum_{|\sigma|=K} 1_{(\sigma \in \mathscr{A}_{L_0}(K))}\Big] + \sum_{k=L_0+1}^{K} \mathbb{E}_0\Big[\sum_{\sigma \in \mathscr{A}_{N,k}} 1_{(x_\sigma < \frac{k-L_0}{K+1})}\Big],\end{aligned} \tag{5.3.57}$$

where the last inequality follows from Markov's inequality. We first compute the second term on the right-hand side of (5.3.57), which is

$$\begin{aligned} &= \sum_{k=L_0+1}^{K} N^k \mathbb{P}\Big[U_1 < \cdots < U_k < \frac{k-L_0}{K+1}\Big] \\ &= \sum_{k=L_0+1}^{K} N^k \Big(\frac{k-L_0}{K+1}\Big)^k \frac{1}{k!}. \end{aligned} \tag{5.3.58}$$

By (5.2.20),

$$\sum_{k=L_0+1}^{K} \mathbb{E}_0\Big[\sum_{\sigma\in\mathscr{A}_{N,k}} 1_{(x_\sigma<\frac{k-L_0}{K+1})}\Big] \le \sum_{k=L_0+1}^{K} \Big(\frac{eN}{K+1}\Big)^k \frac{e^{-L_0}}{2\sqrt{k}} \le c_{12}N^{-3/2}. \tag{5.3.59}$$

The inequality (5.3.57) thus becomes that

$$\begin{aligned} \mathbb{P}_0\Big[Z_{N,eN}\ge 1\Big] &\le \mathbb{E}_0\Big[\sum_{|\sigma|=K} 1_{(\sigma\in\mathscr{A}_{L_0}(K))}\Big] + c_{12}N^{-3/2} \\ &= N^K \mathbb{P}[A_{L_0}(K)] + c_{12}N^{-3/2}, \end{aligned} \tag{5.3.60}$$

where $A_{L_0}(K)$ is defined in (5.3.5). Applying Lemma 5.3.2 yields that

$$\begin{aligned} \mathbb{P}_0\Big[Z_{N,eN}\ge 1\Big] &\le N^K \frac{e^{c_0\sqrt{L_0}}}{K^{3/2}} \frac{e^K}{(K+1)^K} + c_{12}N^{-3/2} \\ &\le c_{13}\frac{e^{c_0\sqrt{2\log K}}}{N^{3/2}} = N^{-3/2+o_N(1)}, \end{aligned} \tag{5.3.61}$$

which completes the proof of (ii) of Theorem 5.1.3. □

5.4 Proof of Theorem 5.1.4

5.4.1 The second moment of $Z_{N,\alpha N}$

Let $m_k(x) := \mathbb{E}_x\big[Z_{N,k}\big]$ for any $k\ge 1$ and $x\in[0,1]$. Recall that

$$m_k(x) = \frac{(1-x)^k N^k}{k!}. \tag{5.4.1}$$

We state the following lemma, concerning the second moment of $Z_{N,\alpha N}$.

Lemma 5.4.1. *For $x\in[0,1)$ fixed and $0<\alpha<2(1-x)$, we have*

$$\lim_{N\to\infty} \frac{\mathbb{E}_x\big[\big(Z_{N,\alpha N}\big)^2\big]}{m_{\alpha N}(x)^2} = \frac{2(1-x)}{2(1-x)-\alpha}. \tag{5.4.2}$$

Proof. By (5.3.36),

$$\begin{aligned}
\mathbb{E}_x\big[\big(Z_{N,k}\big)^2\big] &= m_k(x) + \frac{N-1}{N}\sum_{q=0}^{k-1} N^{2k-q}\mathbb{P}_x(\sigma,\sigma' \in \mathscr{A}_{N,k}\Big||\sigma\wedge\sigma'| = q) \\
&= m_k(x) + \frac{N-1}{N}\sum_{q=0}^{k-1}\frac{((1-x)N)^{2k-q}}{(2k-q)!}\binom{2(k-q)}{(k-q)} \\
&= m_K(x) + m_K(x)^2\frac{N-1}{N}\sum_{q=0}^{k-1} a_k(q,x), \qquad (5.4.3)
\end{aligned}$$

where $a_k(q,x) := \frac{(2k-2q)!k!k!}{[(1-x)N]^q(2k-q)!(k-q)!(k-q)!}$. Note that if $k+1 \leq 2(1-x)N$,

$$a_k(q+1,x) = a_k(q,x)\frac{(k-q)(2k-q)}{2(1-x)N(2k-2q-1)} \leq a_k(q,x), \quad \forall 0 \leq q < k. \qquad (5.4.4)$$

Moreover, for $q \ll \sqrt{k}$ and $k = \alpha N$,

$$a_k(q,x) = \Big(\frac{k}{2(1-x)N}\Big)^q\frac{[(1-\frac{1}{k})\cdots(1-\frac{q-1}{k})]^2}{(1-\frac{q}{2k})\cdots(1-\frac{2q-1}{2k})} = \Big(\frac{\alpha}{2(1-x)}\Big)^q[1+O(\frac{q^2}{k})]. \qquad (5.4.5)$$

Take $q_0 = \lceil\frac{2\log N}{\log(2(1-x))-\log\alpha}\rceil$ so that $\Big(\frac{\alpha}{2(1-x)}\Big)^{q_0} \leq N^{-2}$. It follows from (5.4.4) that

$$\sum_{q=q_0}^{k-1} a_k(q,x) \leq k a_k(q_0,x) \leq c_{13}\alpha N^{-1}, \qquad (5.4.6)$$

which vanished as N goes to infinity. The dominated convergence theorem implies that for $0 < \alpha < 2(1-x)$ and $k = \alpha N$,

$$\lim_{N\to\infty}\sum_{q=0}^{q_0} a_k(q,x) = \sum_{q=0}^{\infty}\Big(\frac{\alpha}{2(1-x)}\Big)^q = \frac{2(1-x)}{2(1-x)-\alpha}. \qquad (5.4.7)$$

Moreover, $1/m_{\alpha N}(x) \to 0$ as N goes to infinity. We thus conclude that for $0 < \alpha < 2(1-x)$,

$$\lim_{N\to\infty}\frac{\mathbb{E}_x\big[\big(Z_{N,\alpha N}\big)^2\big]}{m_{\alpha N}(x)^2} = \frac{2(1-x)}{2(1-x)-\alpha}. \qquad \square$$

As a consequence of Lemma 5.4.1, one sees that $\mathbb{E}_{\lambda/N}\Big[\Big(\frac{Z_{N,N}}{m_N}\Big)^2\Big] \to e^{-2\lambda}$ as $N \to \infty$.

5.4.2 Convergence in law of $Z_{N,N}$

In this subsection, we investigate $Z_{N,N}$. Let $\{\mathscr{F}_k; k \geq 1\}$ denote the natural filtration of the accessibility percolation on N-ary tree, i.e., $\mathscr{F}_k := \sigma\{(\omega, x_\omega); \omega \in T^{(N)}, |\omega| \leq k\}$.

We introduce the following variables :

$$\theta_{N,k}(x) := \mathbb{E}_x\Big[Z_{N,N}|\mathscr{F}_k\Big], \quad \text{and } \widetilde{\theta}_{N,k}(x) := \mathbb{E}_x\Big[Z_{N,N+k}|\mathscr{F}_k\Big].$$

Let $\theta := Z_{N,N}$ for simplicity.

Recall that $m_N = \mathbb{E}_0[\theta] = \frac{N^N}{N!}$. We begin with the following lemma.

Lemma 5.4.2. *As N goes to infinity then k goes to infinity,*

$$\mathscr{L}\Big(\frac{\theta_{N,k}(\lambda/N) - \theta}{m_N}, \mathbb{P}_{\lambda/N}\Big) \to 0. \tag{5.4.8}$$

Proof. We observe that for any $z \in \mathbb{R}$ and $\delta > 0$,

$$\mathbb{P}_x[\theta_{N,k}(x) \leq (z-\delta)m_N|\mathscr{F}_k] - \mathbb{P}_x[|\theta - \theta_{N,k}(x)| \geq m_N\delta|\mathscr{F}_k] \leq \mathbb{P}_x[\theta \leq m_N z|\mathscr{F}_k];$$
$$\mathbb{P}_x[\theta_{N,k}(x) \leq (z+\delta)m_N|\mathscr{F}_k] + \mathbb{P}_x[|\theta - \theta_{N,k}(x)| \geq m_N\delta|\mathscr{F}_k] \geq \mathbb{P}_x[\theta \leq m_N z|\mathscr{F}_k].$$

Note also that

$$\mathbb{P}_x[|\theta - \theta_{N,k}(x)| \geq m_N\delta|\mathscr{F}_k] \leq \frac{\mathrm{Var}_x(\theta|\mathscr{F}_k)}{m_N^2\delta^2}. \tag{5.4.9}$$

Consequently,

$$\mathbb{P}_x[\theta_{N,k}(x) \leq (z-\delta)m_N] - \mathbb{P}_x[\theta \leq m_N z] \leq \mathbb{E}_x\Big[\frac{\mathrm{Var}_x(\theta|\mathscr{F}_k)}{m_N^2\delta^2}\Big];$$
$$\mathbb{P}_x[\theta \leq m_N z] - \mathbb{P}_x[\theta_{N,k}(x) \leq (z+\delta)m_N] \leq \mathbb{E}_x\Big[\frac{\mathrm{Var}_x(\theta|\mathscr{F}_k)}{m_N^2\delta^2}\Big].$$

Thus, it suffices to prove the following convergence.

$$\lim_{k\to\infty}\lim_{N\to\infty} \frac{\mathbb{E}_{\lambda/N}[\mathrm{Var}(\theta|\mathscr{F}_k)]}{m_N^2} = 0. \tag{5.4.10}$$

The branching property yields that

$$\mathrm{Var}_x(\theta|\mathscr{F}_k) = \sum_{\sigma \in \mathscr{A}_{N,k}} v(x_\sigma, N-k), \tag{5.4.11}$$

where $v(y,L) := \mathbb{E}_y[(Z_{N,L})^2] - \mathbb{E}_y[Z_{N,L}]^2$ for any $L \geq 1$. Taking the expectation implies that

$$\mathbb{E}_x[\mathrm{Var}_x(\theta|\mathscr{F}_k)] \;=\; N^k \int_x^1 dy \frac{y^{k-1}}{(k-1)!} v(y, N-k). \tag{5.4.12}$$

By (5.4.3), we have

$$\begin{aligned} v(y,L) &= m_L(y)+m_L(y)^2\frac{N-1}{N}\sum_{q=0}^{L-1}a_L(q,y)-m_L(y)^2 \\ &= m_L(y)+m_L(y)^2\frac{N-1}{N}\sum_{q=1}^{L-1}a_L(q,y)-\frac{m_L(y)^2}{N}. \end{aligned}$$

Plugging it into (5.4.12) yields that

$$\mathbb{E}_x[\mathrm{Var}_x(\theta|\mathscr{F}_k)] \quad = \quad m_N(x)+m_N(x)^2\frac{N-1}{N}\sum_{q=k+1}^{N-1}a_N(q,x)-\frac{1}{N}m_N(x)^2a_N(k,x).$$

It follows from (5.4.5) and (5.4.1) that $\sum_{q=k+1}^{N-1}a_N(q,\lambda/N)\to\frac{1}{2^k}$. Clearly, $m_N(\lambda/N)/m_N=(1-\lambda/N)^N\to e^{-\lambda}$. Therefore,

$$\lim_{N\to\infty}\frac{\mathbb{E}_{\lambda/N}[\mathrm{Var}_{\lambda/N}(\theta|\mathscr{F}_k)]}{m_N^2}=\frac{e^{-2\lambda}}{2^k}, \tag{5.4.13}$$

which vanishes as k goes to infinity. This yields (5.4.10) and completes the proof of Lemma 5.4.2. □

Lemma 5.4.3. *For any $k\geq 0$ fixed, we have*

$$\lim_{N\to\infty}\frac{\mathbb{E}_{\lambda/N}[(\theta_{N,k}(\lambda/N)-\widetilde{\theta}_{N,k}(\lambda/N))^2]}{m_N^2}=0. \tag{5.4.14}$$

Proof. By Jensen's inequality,

$$\Big(\theta_{N,k}(x)-\widetilde{\theta}_{N,k}(x)\Big)^2=\Big(\mathbb{E}_x[Z_{N,N+k}-Z_{N,N}|\mathscr{F}_k]\Big)^2\leq\mathbb{E}_x\Big[\Big(Z_{N,N+k}-Z_{N,N}\Big)^2|\mathscr{F}_k\Big]. \tag{5.4.15}$$

Taking the expectation yields that

$$\mathbb{E}_x\Big[\Big(\theta_{N,k}(x)-\widetilde{\theta}_{N,k}(x)\Big)^2\Big]\leq\mathbb{E}_x\Big[\Big(Z_{N,N+k}-Z_{N,N}\Big)^2\Big], \tag{5.4.16}$$

which, by the Cauchy-Schwarz inequality, is bounded by

$$k\sum_{i=1}^{k}\mathbb{E}_x\Big[\Big(Z_{N,N+i}-Z_{N,N+i-1}\Big)^2\Big]. \tag{5.4.17}$$

Let $L=K+i-1\geq K$. Then,

$$Z_{N,L+1}-Z_{N,L}=\sum_{\sigma\in\mathscr{A}_{N,L}}(y_\sigma-1), \tag{5.4.18}$$

where $y_\sigma := \sum_{|\omega|=L+1} 1_{(\omega_L=\sigma)} 1_{(x_\omega > x_\sigma)}$. It immediately follows that

$$\begin{aligned}\Big(Z_{N,L+1}-Z_{N,L}\Big)^2 &= \sum_{\sigma\in\mathscr{A}_{N,L}} (y_\sigma-1)^2 + \sum_{\sigma\neq\sigma';\sigma,\sigma'\in\mathscr{A}_{N,L}} (y_\sigma-1)(y_{\sigma'}-1) \\ &= \sum_{\sigma\in\mathscr{A}_{N,L}} (y_\sigma-1)^2 + \sum_{q=0}^{L-1}\sum_{|\sigma\wedge\sigma'|=q} 1_{(\sigma,\sigma'\in\mathscr{A}_{N,L})}(y_\sigma-1)(y_{\sigma'}-1), \end{aligned} \tag{5.4.19}$$

where $\sigma\wedge\sigma'$ is, as before, the latest common ancestor of σ and σ'. Note that under $\mathbb{P}_x[\cdot|\mathscr{F}_L]$, y_σ's are independent binomial variables with parameters N and $1-x_\sigma$. Thus, taking $\mathbb{E}_x[\cdot|\mathscr{F}_L]$ on both sides of (5.4.19) yields that

$$\mathbb{E}_x\Big[\Big(Z_{N,L+1}-Z_{N,L}\Big)^2\Big|\mathscr{F}_L\Big] = \Sigma_1+\Sigma_2, \tag{5.4.20}$$

where

$$\Sigma_1 := \sum_{\sigma\in\mathscr{A}_{N,L}} \mathbb{E}_x\Big[(y_\sigma-1)^2\Big|\mathscr{F}_L\Big]; \tag{5.4.21}$$

$$\Sigma_2 := \sum_{q=0}^{L-1}\sum_{|\sigma\wedge\sigma'|=q} 1_{(\sigma,\sigma'\in\mathscr{A}_{N,L})}(N(1-x_\sigma)-1)(N(1-x_{\sigma'})-1). \tag{5.4.22}$$

Obviously, $(y_\sigma-1)^2\le N^2$. Hence,

$$\mathbb{E}_x[\Sigma_1]\le N^2\mathbb{E}_x[Z_{N,L}] = N^2 m_L(x) = o(m_N^2). \tag{5.4.23}$$

Conditioning on the value of $x_{\sigma\wedge\sigma'}$ yields that

$$\begin{aligned}\mathbb{E}_x[\Sigma_2] &= \mathbb{E}_x\Big[\sum_{q=0}^{L-1}\sum_{|\sigma\wedge\sigma'|=q} 1_{(\sigma,\sigma'\in\mathscr{A}_{N,L})}(N(1-x_\sigma)-1)(N(1-x_{\sigma'})-1)\Big] \\ &= \frac{N-1}{N}\sum_{q=0}^{L-1} N^{2L-q}\int_x^1 dy\frac{(y-x)^{q-1}}{(q-1)!}\Big[\int_y^1 dx_\sigma \frac{(x_\sigma-y)^{L-q-1}(N(1-x_\sigma)-1)}{(L-q-1)!}\Big]^2 \\ &= \frac{N-1}{N}\sum_{q=0}^{L-1}(\delta_1(q)-2\delta_2(q)+\delta_3(q)), \end{aligned} \tag{5.4.24}$$

where

$$\begin{aligned}\delta_1(q): &= N^{2L-q}\int_x^1 dy\frac{(y-x)^{q-1}}{(q-1)!}\Big(\frac{N(1-y)^{L-q+1}}{(L-q+1)!}\Big)^2; \\ \delta_2(q): &= N^{2L-q}\int_x^1 dy\frac{(y-x)^{q-1}}{(q-1)!}\Big(\frac{N(1-y)^{L-q+1}}{(L-q+1)!}\times\frac{(1-y)^{L-q}}{(L-q)!}\Big); \\ \delta_3(q): &= N^{2L-q}\int_x^1 dy\frac{(y-x)^{q-1}}{(q-1)!}\Big(\frac{(1-y)^{L-q}}{(L-q)!}\Big)^2. \end{aligned}$$

On the one hand,

$$0 \le \delta_1(q) - 2\delta_2(q) + \delta_3(q) \le 5\delta_3(q), \quad \forall q \ge 0. \tag{5.4.25}$$

On the other hand,

$$\delta_1(q) - 2\delta_2(q) + \delta_3(q) \le \delta_3(q) O(\frac{q^2}{L^2}), \quad \forall q \le O(\log L). \tag{5.4.26}$$

Thus, (5.4.24) becomes

$$\mathbb{E}_x[\Sigma_2] \le \sum_{q=c_{14}\log L}^{L-1} 5\delta_3(q) + \sum_{q=0}^{c_{14}\log L} \delta_3(q) c_{15}(\frac{q^2}{L^2}). \tag{5.4.27}$$

Notice that $\delta_3(q) = m_L^2(x) a_L(q,x)$. Take $x = \lambda/N$ and recall that $L = N + i - 1$. By (5.4.4), for N large enough so that $N + i \le 2(1 - \lambda/N)N$, $a_L(q,x)$ is non-increasing as q increases. It follows that

$$\begin{aligned}
\mathbb{E}_x[\Sigma_2] &\le m_L^2(x)\Big(\sum_{q=c_{14}\log L}^{L-1} 5a_L(q,x) + \sum_{q=0}^{c_{14}\log L} a_L(q,x) c_{15}(\frac{q^2}{L^2})\Big) \\
&\le m_L^2(x)\Big(5La_L(c_{14}\log L, x) + c_{15}\frac{(c_{14}\log L)^3}{L^2} a_L(0,x)\Big).
\end{aligned}$$

Note that $a_L(0,x) = 1$. By (5.4.5), $a_L(c_{14}\log L, x) = \left(\frac{L}{2(1-x)N}\right)^{c_{14}\log L}[1 + O(\frac{(\log L)^2}{L})]$. We can choose a suitable c_{14} so that $a_L(c_{14}\log L, x) = o(N^{-1})$. As a result,

$$\mathbb{E}_x[\Sigma_2] = m_L^2(x) o_N(1) = m_N^2 o_N(1). \tag{5.4.28}$$

We return to (5.4.20). Combining (5.4.23) with (5.4.28) implies that

$$\mathbb{E}_{\lambda/N}\Big[\frac{\Big(Z_{L+1}^{(N)} - Z_L^{(N)}\Big)^2}{m_N^2}\Big] = o_N(1). \tag{5.4.29}$$

Therefore, for any $k \ge 1$ fixed, we have

$$\lim_{N\to\infty} \mathbb{E}_{\lambda/N}[\frac{(\theta_{N,k}(\lambda/N) - \widetilde{\theta}_{N,k}(\lambda/N))^2}{m_N^2}] = 0. \qquad \square$$

By considering the variables $\widetilde{\theta}_{N,k}(x)$, we will prove the convergence in law in Theorem 5.1.4 as follows.

Proof of Theorem 5.1.4. In view of Lemmas 5.4.2 and 5.4.3, we only need to prove that the distribution $\mathscr{L}\Big(\frac{\widetilde{\theta}_{N,k}(\lambda/N)}{m_N},\ \mathbb{P}_{\lambda/N}\Big)$ converges weakly to an exponential variable of mean $e^{-\lambda}$, as N goes to infinity then k goes to infinity.

Clearly, $\widetilde{\theta}_{N,0}(x) = m_N(1-x)^N$ with $m_N = \frac{N^N}{N!}$. Define for any $k \geq 0$ and $\mu \geq 0$,

$$G_k(\mu,x,N) := \mathbb{E}_x\Big[\exp\{-\mu\widetilde{\theta}_{N,k}(x)/m_N\}\Big], \tag{5.4.30}$$

which is the Laplace transform of $\frac{\widetilde{\theta}_{N,k}(x)}{m_N}$.

It is immediate that $G_0(\mu,x,N) = \exp\{-\mu(1-x)^N\}$. Recursively,

$$\widetilde{\theta}_{N,k+1}(x) = \mathbb{E}_x\big[Z_{N,N+k+1}\big|\mathscr{F}_{k+1}\big] = \sum_{\sigma\in\mathscr{A}_{N,1}} \mathbb{E}_{x_\sigma}\Big[Z_{N,N+k}\Big|\mathscr{F}_k\Big] = \sum_{|\sigma|=1} 1_{(x_\sigma>x)}\widetilde{\theta}_{N,k}(x_\sigma), \tag{5.4.31}$$

where for $|\sigma|=1$, $1_{(x_\sigma>x)}\widetilde{\theta}_{N,k}(x_\sigma)$ are i.i.d. It follows that

$$G_{k+1}(\mu,x,N) = \Big[x + \int_x^1 dy G_k(\mu,y,N)\Big]^N. \tag{5.4.32}$$

We define for λ, $\mu > 0$,

$$Q_0(\mu,\lambda) := \exp\{-\mu e^{-\lambda}\}; \tag{5.4.33}$$

$$Q_{k+1}(\mu,\lambda) := \exp\Big\{-\int_\lambda^\infty \big(1-Q_k(\mu,y)\big)dy\Big\}, \quad \forall k \geq 0. \tag{5.4.34}$$

Clearly, $\lim_{N\to\infty} G_0(\mu,\frac{\lambda}{N},N) = Q_0(\mu,\lambda)$. We are going to prove that for any $k \geq 0$,

$$\lim_{N\to\infty} G_k(\mu,\frac{\lambda}{N},N) = Q_k(\mu,\lambda). \tag{5.4.35}$$

Suppose that (5.4.35) holds for $k \geq 0$. By a change of variables, (5.4.32) becomes that

$$G_{k+1}(\mu,\frac{\lambda}{N},N) = \Big[1 - \int_\lambda^N dy\Big(1-G_k(\mu,\frac{y}{N},N)\Big)\Big]^N. \tag{5.4.36}$$

Because $1-e^{-z} \leq z$ for all $z \in \mathbb{R}$, (5.4.30) gives that

$$0 \leq 1-G_k(\mu,\frac{y}{N},N) \leq \mu\mathbb{E}_{\frac{y}{N}}[Z_{N,N+k}]/m_N = \frac{\mu}{m_N}\frac{[N(1-y/N)]^{N+k}}{(N+k)!} \leq \mu e^{-y}. \tag{5.4.37}$$

The dominated convergence theorem implies that

$$\int_\lambda^N dy\Big(1-G_k(\mu,\frac{y}{N},N)\Big) \xrightarrow{N\to\infty} \int_\lambda^\infty dy\Big(1-Q_k(\mu,y)\Big). \tag{5.4.38}$$

It follows that $\lim_{N\to\infty} G_{k+1}(\mu,\frac{\lambda}{N},N) = Q_{k+1}(\mu,X)$. By induction, we conclude (5.4.35) for any $k \geq 0$.

We write $Q_k(\mu,\lambda) = F_k(\mu e^{-\lambda})$ for all $k \geq 0$. We check that

$$F_{k+1}(z) = \exp\Big\{-\int_0^z \frac{1-F_k(u)}{u}du\Big\}, \qquad F_0(z) = e^{-z}. \tag{5.4.39}$$

Define $\Delta_k(z)$ for $z > -1$ and $z \neq 0$ by

$$\Delta_k(z) := 2^k \frac{(1+z)^3}{z^2}\Big[\frac{1}{1+z} - F_k(z)\Big]. \tag{5.4.40}$$

Then we claim that there exists a constant M such that for all $k \geq 0$,

$$0 \leq \Delta_k(z) \leq M, \quad \forall z > -1. \tag{5.4.41}$$

Indeed, for $k = 0$,

$$\Delta_0(z) = \frac{(1+z)^3}{z^2}\Big[\frac{1}{1+z} - e^{-z}\Big], \tag{5.4.42}$$

which is nonnegative for $z > -1$, because $e^z \geq 1+z$. Moreover, since $\lim_{z\to 0}\Delta_0(z) = 1/(2e)$, define $\Delta_0(0) := \frac{1}{2e}$ so that $\Delta_0(z)$ is continuous in $(-1,\infty)$, and that both $\lim_{z\downarrow -1}\Delta_0(z)$ and $\lim_{z\uparrow\infty}\Delta_0(z)$ exist and are bounded. Hence, there exists $M \in (0,\infty)$ such that

$$0 \leq \Delta_0(z) \leq M, \quad \forall z > -1. \tag{5.4.43}$$

Assume now that (5.4.41) holds at order k. In view of (5.4.39) and (5.4.40),

$$F_{k+1}(z) = \frac{1}{1+z}\exp\Big\{-\int_0^z \frac{u}{(1+u)^3}\frac{\Delta_k(u)}{2^k}du\Big\}. \tag{5.4.44}$$

This leads to

$$\frac{1}{1+z} \geq F_{k+1}(z) \geq \frac{1}{1+z}\Big[1 - \int_0^z \frac{u}{(1+u)^3}\frac{M}{2^k}du\Big] = \frac{1}{1+z}\Big[1 - \frac{M}{2^k}\frac{z^2}{2(1+z)^2}\Big].$$

This implies that (5.4.41) holds for $k+1$. In view of (5.4.40) and (5.4.41), we check that

$$\lim_{k\to\infty} F_k(z) = \frac{1}{1+z}, \text{ for } z > -1. \tag{5.4.45}$$

Recall that $Q_k(\mu,\lambda) = F_k(\mu e^{-\lambda})$. Going back to (5.4.35), we let k go to infinity for both sides and obtain that for any $\lambda > 0$ fixed,

$$\lim_{k\to\infty}\lim_{N\to\infty}\mathbb{E}_{\lambda/N}\Big[e^{-\mu\frac{\widetilde{\theta}_{N,k(\lambda/N)}}{m_N}}\Big] = \lim_{k\to\infty}\lim_{N\to\infty} G_k(\mu,\frac{\lambda}{N},N) = \frac{1}{1+\mu e^{-\lambda}}, \tag{5.4.46}$$

which is the Laplace transform of an exponential variable of mean $e^{-\lambda}$. Therefore, we deduce that as $N \to \infty$,

$$\mathscr{L}\left(\frac{Z_{N,N}}{m_N}, \mathbb{P}_{\lambda/N}\right) \to e^{-\lambda} \times W, \tag{5.4.47}$$

where W is an exponential variable with mean 1. $\square$

An analogous argument implies that for $0 < \alpha < 1$, started from $x = 1 - \alpha + \frac{\lambda}{N}$, $\mathscr{L}\left(\frac{Z_{N,\alpha N}}{m_{\alpha N}(1-\alpha)}, \mathbb{P}_x\right)$ converges to an exponential distribution of mean $e^{-\lambda}$.

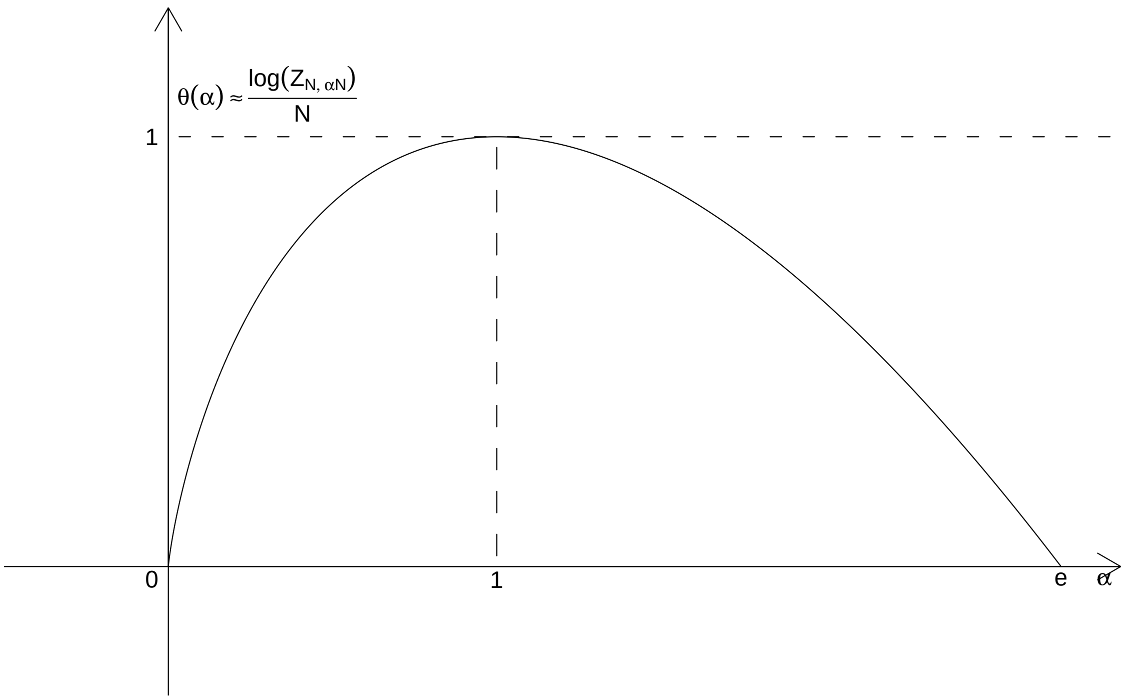

FIGURE 5.1 – The curve of $\alpha \mapsto \theta(\alpha) = \alpha(1 - \log \alpha)$.

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